Midterm, mechanical vibration

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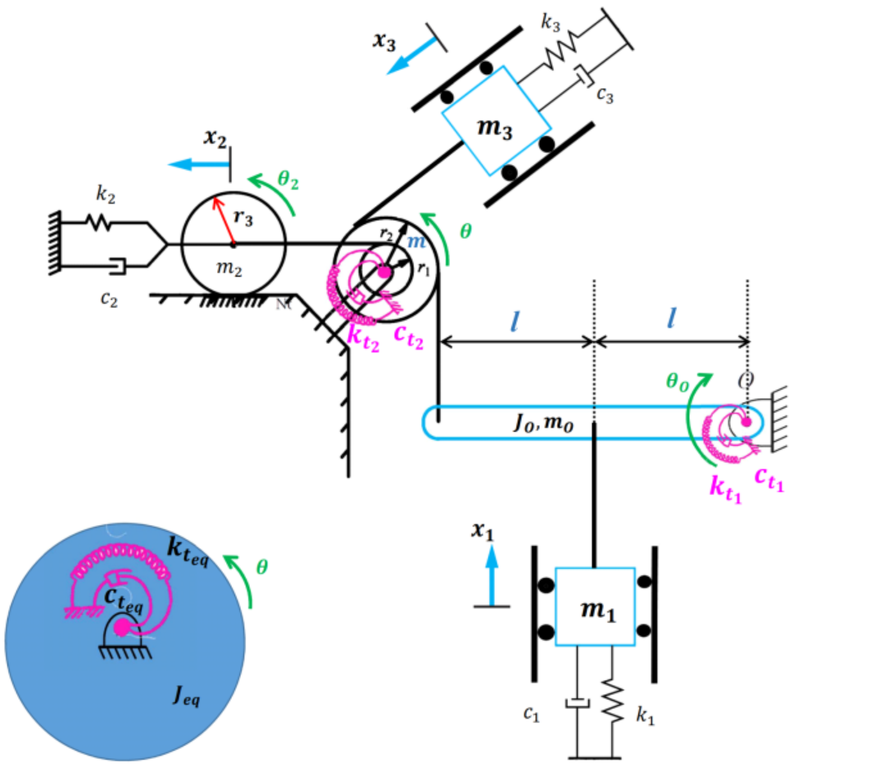
ذکر تمامی جزئیات زیر نمره دارد.

Determine equivalent mass moment of inertia, equivalent spring and equivalent damper. Write equation of motion and solve it for initial conditions.



k_1	$1000\ \frac{N}{m}$
k_{t_1}	$100\ \frac{N.m}{rad}$
k_2	$2000\ \frac{N}{m}$
k_{t_2}	$200\ \frac{N.m}{rad}$
k_3	$500\ \frac{N}{m}$
c_1	$10\ \frac{N.s}{m}$
c_{t_1}	$5\ \frac{N.m.s}{rad}$
c_2	$4\ \frac{N.s}{m}$
c_{t_2}	$2\ \frac{N.m.s}{rad}$
c_3	$2\ \frac{N.s}{m}$
m_1	2 kg
m_2	6 kg
m_3	1 kg
m	8 kg
J_O	20 $kg.m^2$
r_1	1 m
r_2	2 m
r_3	3 m
l	10 m
θ_0	0.1 rad
$\dot{\theta}_0$	$3\ \frac{rad}{s}$

Determine equialent mass moment of inertia

$\boxed{\text{Kinetic Energy}}$

$J_2 = \frac{1}{2} m_2 r_3^2$

$\frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} J_2 \dot{\theta}_2^2$

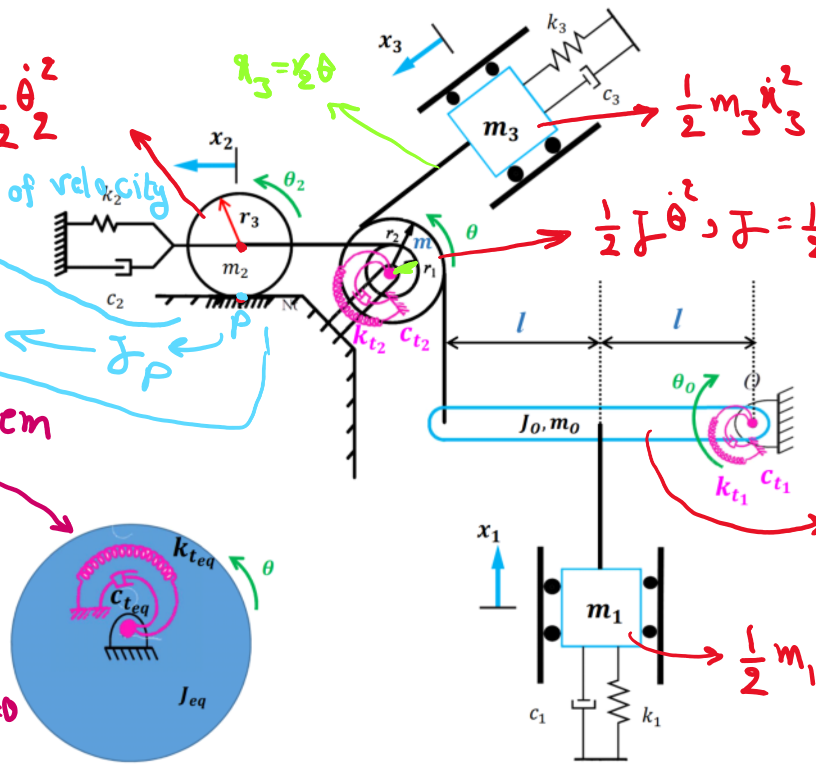
$x_3 = r_2 \theta$

$\frac{1}{2} m_3 \dot{x}_3^2$

P: Instant center of velocity

$\frac{1}{2} J \dot{\theta}^2, \quad J = \frac{1}{2} m r_2^2$

$\frac{1}{2} J_P \dot{\theta}_2^2 \leftarrow J_P \leftarrow P$



$x_2 = r_1 \theta$

$\underbrace{2l\theta_0}_{=2x_1} = r_2 \theta$

Equivalent system

$\frac{1}{2} J_0 \dot{\theta}_0^2$

Equation of motion

$\frac{1}{2} m_1 \dot{x}_1^2$

$\boxed{l\theta_0 = x_1}$

$J_{eq}\ddot{\theta} + (C_t)_{eq}\dot{\theta} + (k_t)_{eq}\theta = 0$

Kinetic energy of system

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} J_0 \dot{\theta}_0^2 + \frac{1}{2} J \dot{\theta}^2 + \underbrace{\frac{1}{2} J_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 \dot{x}_2^2}_{= \frac{1}{2} J_p \dot{\theta}_2^2} + \frac{1}{2} m_3 \dot{x}_3^2 \quad (1)$$

$$T_{eq} = \frac{1}{2} J_{eq} \dot{\theta}^2$$

$$\begin{cases} l\theta_0 = x_1 \\ r_2\theta = 2l\theta_0 = 2x_1 \\ x_2 = r_1\theta = r_3\theta_2 \\ x_3 = r_2\theta \end{cases} \begin{matrix} \Rightarrow x_1 = \frac{r_2}{2}\theta \, , \; \theta_0 = \frac{r_2}{2L}\theta \\ \Rightarrow \theta_2 = \frac{r_1}{r_3}\theta \end{matrix}$$

$$\begin{cases} x_1 = \frac{r_2}{2}\theta \\ x_2 = r_1\theta \\ x_3 = r_2\theta \\ \theta_0 = \frac{r_2}{2L}\theta \\ \theta_2 = \frac{r_1}{r_3}\theta \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = \frac{r_2}{2}\dot{\theta} \\ \dot{x}_2 = r_1\dot{\theta} \\ \dot{x}_3 = r_2\dot{\theta} \\ \dot{\theta}_0 = \frac{r_2}{2L}\dot{\theta} \\ \dot{\theta}_2 = \frac{r_1}{r_3}\dot{\theta} \end{cases} \quad (2) \text{ and } \begin{cases} dx_1 = \frac{r_2}{2} d\theta \\ dx_2 = r_1 d\theta \\ dx_3 = r_2 d\theta \\ d\theta_0 = \frac{r_2}{2L} d\theta \\ d\theta_2 = \frac{r_1}{r_3} d\theta \end{cases}$$

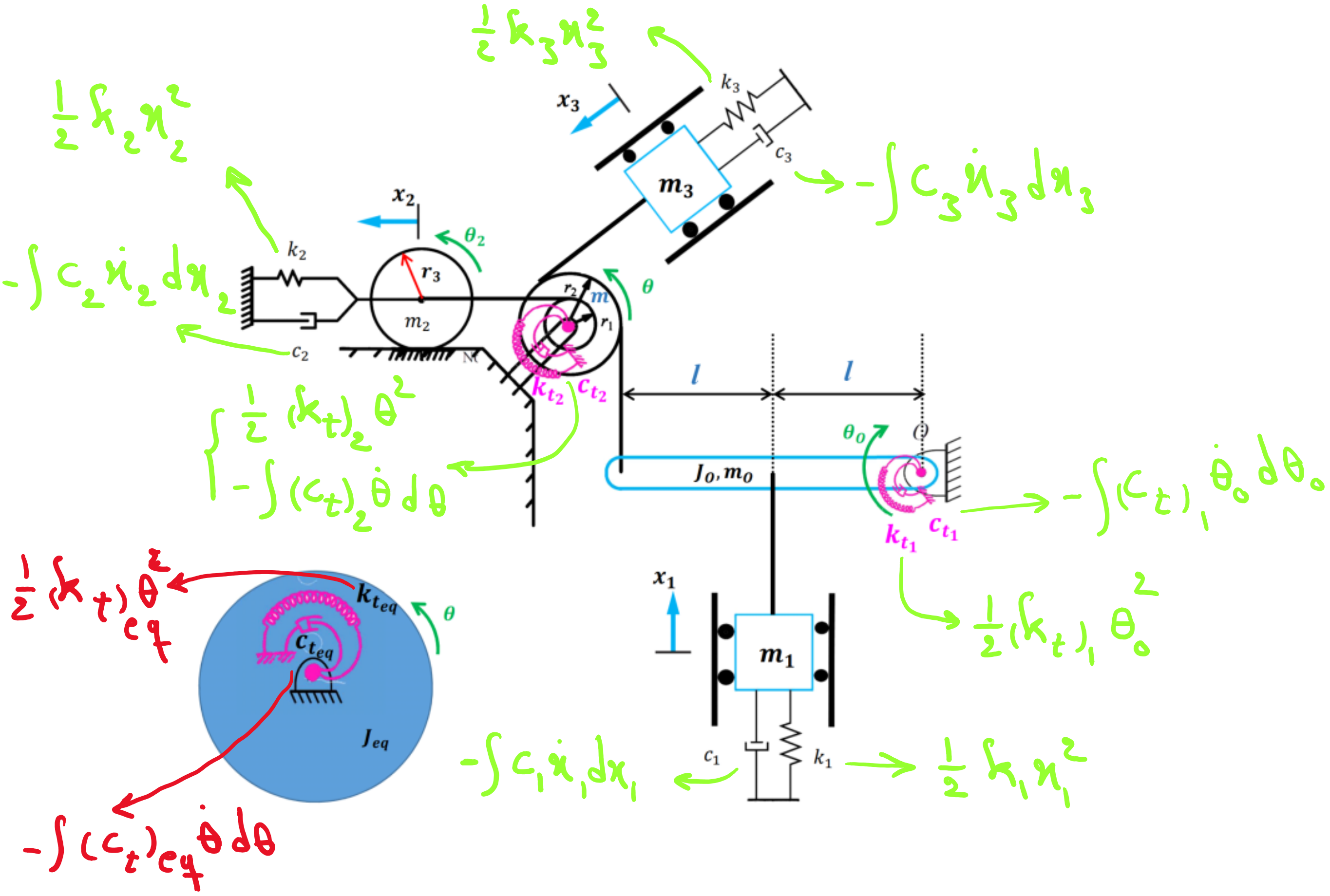
(2) into (1)

$$\frac{1}{2} J_{eq} \dot{\theta}^2 = \frac{1}{2} m_1 \left(\frac{r_2}{2}\dot{\theta}\right)^2 + \frac{1}{2} J_0 \left(\frac{r_2}{2L}\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{2} m r_2^2\right)\dot{\theta}^2 + \frac{1}{2}\left(\frac{1}{2} m_2 r_3^2\right)\left(\frac{r_1}{r_3}\dot{\theta}\right)^2$$
$$+ \frac{1}{2} m_2 (r_1\dot{\theta})^2 + \frac{1}{2} m_3 (r_2\dot{\theta})^2 \quad \Rightarrow$$

$$J_{eq} = \frac{1}{4} m_1 r_1^2 + \frac{1}{4}\left(\frac{r_2}{L}\right)^2 J_0 + \frac{1}{2} m r_2^2 + \underbrace{\underbrace{\frac{1}{2} m_2 r_3^2 \left(\frac{r_1}{r_3}\right)^2}_{\frac{1}{2} m_2 r_1^2} + m_2 r_1^2}_{\frac{3}{2} m_2 r_1^2} + m_3 r_2^2$$

$$\boxed{J_{eq}=\frac{1}{4}m_1r_2^2+\frac{1}{4}\left(\frac{r_2}{l}\right)^2J_0+\frac{1}{2}mr_2^2+\frac{3}{2}m_2r_1^2+m_3r_2^2} \quad (3)$$

Equivalent spring $\longrightarrow$ Total potential energy of system



$$P_T=\frac{1}{2}k_1x_1^2+\frac{1}{2}k_2x_2^2+\frac{1}{2}k_3x_3^2+\frac{1}{2}(k_t)_1\theta_0^2+\frac{1}{2}(k_t)_2\theta^2$$

$$\frac{1}{2}(k_t)_{eq}\theta^2=\frac{1}{2}k_1\left(\frac{r_2}{2}\theta\right)^2+\frac{1}{2}k_2(r_1\theta)^2+\frac{1}{2}k_3(r_2\theta)^2+\frac{1}{2}(k_t)_1\left(\frac{r_2}{2l}\theta\right)^2+\frac{1}{2}(k_t)_2\theta^2$$

$$\boxed{(k_t)_{eq}=\frac{1}{4}k_1r_2^2+k_2r_1^2+k_3r_2^2+\frac{1}{4}\left(\frac{r_2}{l}\right)^2(k_t)_1+(k_t)_2} \quad (4)$$

Total dissipated energy of system

$-\int c\dot{x}dx-\int c\dot{x}dx-\int c\dot{x}dx-\int(c)\dot{\theta}d\theta-\int(c)\dot{\theta}d\theta$

$$-\int c_1\dot{x}_1 dx_1-\int c_2\dot{x}_2 dx_2-\int c_3\dot{x}_3 dx_3-\int (c_t)_1\dot{\theta}_0 d\theta_0-\int (c_t)_2\dot{\theta}\,d\theta$$

Dissipated energy of equivalent system $= -\int (c_t)_{eq}\dot{\theta}\,d\theta$

$$-\int (c_t)_{eq}\dot{\theta}\,d\theta = -\int c_1\left(\frac{r_2}{2}\dot{\theta}\right)\left(\frac{r_2}{2}d\theta\right)-\int c_2(r_1\dot{\theta})(r_1 d\theta)-\int c_3(r_2\dot{\theta})(r_2 d\theta)$$

$$-\int (c_t)_1\left(\frac{r_2}{2l}\dot{\theta}\right)\left(\frac{r_2}{2l}d\theta\right)-\int (c_t)_2\dot{\theta}\,d\theta$$

$$\boxed{(c_t)_{eq}=\frac{1}{4}c_1 r_2^2+c_2 r_1^2+c_3 r_2^2+\frac{1}{4}(c_t)_1\left(\frac{r_2}{l}\right)^2+(c_t)_2} \quad (5)$$

Equation of motion: $J_{eq}\ddot{\theta}+(c_t)_{eq}\dot{\theta}+(k_t)_{eq}\theta=0$ (6)

$\theta_{(t)}=\theta=Ce^{st} \Rightarrow \dot{\theta}=Cse^{st} \Rightarrow \ddot{\theta}=Cs^2e^{st}$ (7) $\Rightarrow$

(7) into (6) $\Rightarrow J_{eq}(Cs^2e^{st})+(c_t)_{eq}(Cse^{st})+(k_t)_{eq}(Ce^{st})=0 \Rightarrow$

$$\boxed{J_{eq}s^2+(c_t)_{eq}s+(k_t)_{eq}=0} \quad (8)$$

$$s_1, s_2=\frac{-(c_t)_{eq}\pm\sqrt{[(c_t)_{eq}]^2-4\times J_{eq}\times(k_t)_{eq}}}{2J_{eq}}$$

$$\omega_n=\sqrt{\frac{(k_t)_{eq}}{J_{eq}}}, \quad (c_t)_c=2J_{eq}\omega_n=2\sqrt{(k_t)_{eq}J_{eq}}$$

$$\zeta=\frac{(c_t)_{eq}}{(c_t)_c}<1$$

If $[(c_t)_{eq}]^2 - 4J_{eq}(k_t)_{eq} < 0 \Rightarrow$

$$s_1, s_2 = \frac{-(c_t)_{eq} \pm i\sqrt{4J_{eq}(k_t)_{eq} - [(c_t)_{eq}]^2}}{2J_{eq}}$$

$$= -\frac{(c_t)_{eq}}{2J_{eq}} \pm i\sqrt{\frac{4J_{eq}(k_t)_{eq} - [(c_t)_{eq}]^2}{4J_{eq}^2}}$$

$$= \underbrace{-\frac{(c_t)_{eq}}{2J_{eq}}} \pm i\sqrt{\underbrace{\frac{(k_t)_{eq}}{J_{eq}}}_{\omega_n^2} - \underbrace{\left[\frac{(c_t)_{eq}}{2J_{eq}}\right]^2}}$$

$$\omega_n = \sqrt{\frac{(k_t)_{eq}}{J_{eq}}}$$

$$(C_t)_c = 2J_{eq}\omega_n \Rightarrow \zeta = \frac{(C_t)_{eq}}{(C_t)_c} = \frac{(C_t)_{eq}}{2J_{eq}\omega_n} \Rightarrow \boxed{\frac{(C_t)_{eq}}{2J_{eq}} = \zeta\omega_n}$$

$$s_1, s_2 = -\zeta\omega_n \pm i\sqrt{\omega_n^2 - (\zeta\omega_n)^2} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \Rightarrow \boxed{s_1, s_2 = -\zeta\omega_n \pm i\omega_d} \quad (9)$$

(9) into (7) $\Rightarrow \theta = C_1 e^{s_1 t} + C_2 e^{s_2 t} \Rightarrow$

⑨ into ⑦ $\Rightarrow \theta = C_1 e^{\cdot} + C_2 e^{\cdot} \Rightarrow$

$$\theta = C_1 e^{(-\zeta\omega_n + i\omega_d)t} + C_2 e^{(-\zeta\omega_n + i\omega_d)t}$$

$$= C_1 e^{-\zeta\omega_n t} e^{i\omega_d t} + C_2 e^{-\zeta\omega_n t} e^{-i\omega_d t} = e^{-\zeta\omega_n t}\left[C_1 e^{i\omega_d t} + C_2 e^{-i\omega_d t}\right] \quad (10)$$

$$e^{\pm i\alpha} = \cos\alpha \pm i\sin\alpha$$

$$\theta = e^{-\zeta\omega_n t}\left[C_1(\cos\omega_d t + i\sin\omega_d t) + C_2(\cos\omega_d t - i\sin\omega_d t)\right]$$

$$= e^{-\zeta\omega_n t}\left[\underbrace{(C_1 + C_2)}_{A_1}\cos\omega_d t + i\underbrace{(C_1 - C_2)}_{A_2}\sin\omega_d t\right] \Rightarrow$$

$$\boxed{\theta = e^{-\zeta\omega_n t}\left[A_1\cos\omega_d t + A_2\sin\omega_d t\right]} \quad (11)$$

Calculate unknowns (A_1, A_2) using initial conditions.

$$\text{I.C.}\begin{cases}\theta_{(0)} = \theta_0 & t = 0\\ \dot{\theta}_{(0)} = \dot{\theta}_0 & t = 0\end{cases}$$

$$(11) \Rightarrow \boxed{\dot{\theta} = -\zeta\omega_n e^{-\zeta\omega_n t}\left[A_1\cos\omega_d t + A_2\sin\omega_d t\right] + \omega_d e^{-\zeta\omega_n t}\left[-A_1\sin\omega_d t + A_2\cos\omega_d t\right]} \quad (12)$$

(11) 0

$$\theta\Big|_{t=0} = \theta_0 \overset{(11)}{=\!=} \underbrace{e^{0}}_{1}\big[\underbrace{A_1\cos 0}_{A_1} + \underbrace{A_2 \sin 0}_{0}\big] \Rightarrow \boxed{A_1 = \theta_0} \quad (13)$$

$$\dot\theta\Big|_{t=0} = \dot\theta_0 \overset{(12)}{=\!=} -\zeta\omega_n e^{0}\big[\underbrace{A_1\cos 0}_{A_1} + \underbrace{A_2\sin 0}_{0}\big] + \omega_d \underbrace{e^{0}}_{1}\big[-\underbrace{A_1\sin 0}_{0} + \underbrace{A_2\cos 0}_{A_2}\big]$$

$$\dot\theta_0 = -\zeta\omega_n A_1 + \omega_d A_2 \Rightarrow A_2 = \frac{\dot\theta_0 + \zeta\omega_n A_1}{\omega_d} \quad (14)$$

(13) into (14) $\Rightarrow$

$$\boxed{A_2 = \frac{\dot\theta_0 + \zeta\omega_n\theta_0}{\omega_d}} \quad (15)$$

(13), (15) into (11) $\Rightarrow$

$$\boxed{\theta = e^{-\zeta\omega_n t}\left[\theta_0\cos\omega_d t + \frac{\dot\theta_0 + \zeta\omega_n\theta_0}{\omega_d}\sin\omega_d t\right]} \quad (16)$$

According to the Table

$J_{eq} = 31.2 \ (kg.m^2)$

$(C_t)_{eq} = 24.05 \ \left(\frac{N.m.s}{rad}\right)$

$(k_t)_{eq} = 5201 \ \left(\frac{N.m}{rad}\right)$

$\omega_n = 12.9112$

$(C_t)_c = 805.6580$

$\zeta = 0.0299$

$\omega_d = 12.9054$