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# 2.7 Bounded and Continuous Linear Operators

**2.7-1 Definition (Bounded linear operator).** Let $X$ and $Y$ be normed spaces and $T: \mathfrak{D}(T) \longrightarrow Y$ a linear operator, where $\mathfrak{D}(T) \subset X$. The operator $T$ is said to be *bounded* if there is a real number $c$ such that for all $x \in \mathfrak{D}(T)$,

$$\|Tx\| \leqq c\|x\|. \tag{1}$$

∎

*Warning.* Note that our present use of the word "bounded" is different from that in calculus, where a bounded function is one whose range is a bounded set.

(2) $$\|T\| = \sup_{\substack{x \in \mathscr{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}.$$

$\|T\|$ is called the **norm** of the operator $T$. If $\mathscr{D}(T) = \{0\}$, we define $\|T\| = 0$; in this (relatively uninteresting) case, $T = 0$ since $T0 = 0$ by (3), Sec. 2.6.

Note that (1) with $c = \|T\|$ is

(3) $$\|Tx\| \leqq \|T\| \|x\|.$$

**2.7-2 Lemma (Norm).** *Let $T$ be a bounded linear operator as defined in 2.7-1. Then:*

**(a)** *An alternative formula for the norm of $T$ is*

(4) $$\|T\| = \sup_{\substack{x \in \mathscr{D}(T) \\ \|x\| = 1}} \|Tx\|.$$

**(b)** *The norm defined by (2) satisfies* (N1) *to* (N4) *in Sec.* 2.2.

*Proof.* **(a)** We write $\|x\|=a$ and set $y=(1/a)x$, where $x\neq 0$. Then $\|y\|=\|x\|/a=1$, and since $T$ is linear, (2) gives

$$\|T\|=\sup_{\substack{x\in\mathscr{D}(T)\\ x\neq 0}}\frac{1}{a}\|Tx\|=\sup_{\substack{x\in\mathscr{D}(T)\\ x\neq 0}}\left\|T\left(\frac{1}{a}x\right)\right\|=\sup_{\substack{y\in\mathscr{D}(T)\\ \|y\|=1}}\|Ty\|.$$

Writing $x$ for $y$ on the right, we have (4).

**(b)** (N1) is obvious, and so is $\|0\|=0$. From $\|T\|=0$ we have $Tx=0$ for all $x\in\mathscr{D}(T)$, so that $T=0$. Hence (N2) holds. Furthermore, (N3) is obtained from

$$\sup_{\|x\|=1}\|\alpha Tx\|=\sup_{\|x\|=1}|\alpha|\,\|Tx\|=|\alpha|\sup_{\|x\|=1}\|Tx\|$$

where $x\in\mathscr{D}(T)$. Finally, (N4) follows from

$$\sup_{\|x\|=1}\|(T_1+T_2)x\|=\sup_{\|x\|=1}\|T_1x+T_2x\|\leqq\sup_{\|x\|=1}\|T_1x\|+\sup_{\|x\|=1}\|T_2x\|;$$

here, $x\in\mathscr{D}(T)$. ∎

## Examples

**2.7-3 Identity operator.** The identity operator $I: X \longrightarrow X$ on a normed space $X \neq \{0\}$ is bounded and has norm $\|I\| = 1$. Cf. 2.6-2.

**2.7-4 Zero operator.** The zero operator $0: X \longrightarrow Y$ on a normed space $X$ is bounded and has norm $\|0\| = 0$. Cf. 2.6-3.

**2.7-5 Differentiation operator.** Let $X$ be the normed space of all polynomials on $J = [0, 1]$ with norm given $\|x\| = \max |x(t)|$, $t \in J$. A differentiation operator $T$ is defined on $X$ by

$$Tx(t) = x'(t)$$

where the prime denotes differentiation with respect to $t$. This operator is linear but not bounded. Indeed, let $x_n(t) = t^n$, where $n \in \mathbf{N}$. Then $\|x_n\| = 1$ and

$$Tx_n(t) = x_n'(t) = nt^{n-1}$$

so that $\|Tx_n\| = n$ and $\|Tx_n\|/\|x_n\| = n$. Since $n \in \mathbf{N}$ is arbitrary, this shows that there is no fixed number $c$ such that $\|Tx_n\|/\|x_n\| \leqq c$. From this and (1) we conclude that $T$ is not bounded.

**2.7-6 Integral operator.** We can define an integral operator $T\colon C[0,1] \longrightarrow C[0,1]$ by

$$y = Tx \qquad \text{where} \qquad y(t) = \int_0^1 k(t, \tau) x(\tau)\, d\tau.$$

Here $k$ is a given function, which is called the *kernel* of $T$ and is assumed to be continuous on the closed square $G = J \times J$ in the $t\tau$-plane, where $J = [0, 1]$. This operator is linear.

$T$ is bounded.

To prove this, we first note that the continuity of $k$ on the closed square implies that $k$ is bounded, say, $|k(t, \tau)| \leqq k_0$ for all $(t, \tau) \in G$, where $k_0$ is a real number. Furthermore,

$$|x(t)| \leqq \max_{t \in J} |x(t)| = \|x\|.$$

Hence

$$\begin{aligned} \|y\| = \|Tx\| &= \max_{t \in J} \left| \int_0^1 k(t, \tau) x(\tau)\, d\tau \right| \\ &\leqq \max_{t \in J} \int_0^1 |k(t, \tau)|\, |x(\tau)|\, d\tau \\ &\leqq k_0 \|x\|. \end{aligned}$$

The result is $\|Tx\| \leqq k_0 \|x\|$. This is (1) with $c = k_0$. Hence $T$ is bounded.

**2.7-7 Matrix.** A real matrix $A=(\alpha_{jk})$ with $r$ rows and $n$ columns defines an operator $T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^r$ by means of

$$y = Ax \tag{5}$$

where $x=(\xi_j)$ and $y=(\eta_j)$ are column vectors with $n$ and $r$ components, respectively, and we used matrix multiplication, as in 2.6-8. In terms of components, (5) becomes

$$\eta_j = \sum_{k=1}^{n} \alpha_{jk}\xi_k \qquad (j=1, \cdots, r). \tag{5'}$$

$T$ is linear because matrix multiplication is a linear operation.

*$T$ is bounded.*

To prove this, we first remember from 2.2-2 that the norm on $\mathbf{R}^n$ is given by

$$\|x\| = \left(\sum_{m=1}^{n} \xi_m{}^2\right)^{1/2};$$

similarly for $y \in \mathbf{R}^r$. From (5') and the Cauchy-Schwarz inequality (11) in Sec. 1.2 we thus obtain

$$\begin{aligned}\|Tx\|^2 = \sum_{j=1}^{r} \eta_j{}^2 &= \sum_{j=1}^{r} \left[\sum_{k=1}^{n} \alpha_{jk}\xi_k\right]^2 \\ &\leqq \sum_{j=1}^{r} \left[\left(\sum_{k=1}^{n} \alpha_{jk}{}^2\right)^{1/2} \left(\sum_{m=1}^{n} \xi_m{}^2\right)^{1/2}\right]^2 \\ &= \|x\|^2 \sum_{j=1}^{r} \sum_{k=1}^{n} \alpha_{jk}{}^2.\end{aligned}$$

Noting that the double sum in the last line does not depend on $x$, we can write our result in the form

$$\|Tx\|^2 \leqq c^2\|x\|^2 \qquad \text{where} \qquad c^2 = \sum_{j=1}^{r} \sum_{k=1}^{n} \alpha_{jk}{}^2.$$

This gives (1) and completes the proof that $T$ is bounded. ∎

**2.7-8 Theorem (Finite dimension).** *If a normed space $X$ is finite dimensional, then every linear operator on $X$ is bounded.*

*Proof.* Let $\dim X = n$ and $\{e_1, \cdots, e_n\}$ a basis for $X$. We take any $x = \sum \xi_j e_j$ and consider any linear operator $T$ on $X$. Since $T$ is linear,

$$\|Tx\| = \left\|\sum \xi_j Te_j\right\| \leqq \sum |\xi_j| \, \|Te_j\| \leqq \max_k \|Te_k\| \sum |\xi_j|$$

(summations from 1 to $n$). To the last sum we apply Lemma 2.4-1 with $\alpha_j = \xi_j$ and $x_j = e_j$. Then we obtain

$$\sum |\xi_j| \leqq \frac{1}{c} \left\|\sum \xi_j e_j\right\| = \frac{1}{c} \|x\|.$$

Together,

$$\|Tx\| \leqq \gamma \|x\| \qquad \text{where} \qquad \gamma = \frac{1}{c} \max_k \|Te_k\|.$$

From this and (1) we see that $T$ is bounded. ∎

operator $T$ is *continuous at an* $x_0 \in \mathscr{D}(T)$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|Tx - Tx_0\| < \varepsilon \qquad \text{for all } x \in \mathscr{D}(T) \text{ satisfying} \qquad \|x - x_0\| < \delta.$$

$T$ is *continuous* if $T$ is continuous at every $x \in \mathscr{D}(T)$.

It is a fundamental fact that for a *linear* operator, continuity and boundedness become equivalent concepts.

**2.7-9 Theorem (Continuity and boundedness).** *Let $T: \mathscr{D}(T) \longrightarrow Y$ be a linear*[7] *operator, where $\mathscr{D}(T) \subset X$ and $X$, $Y$ are normed spaces. Then:*

**(a)** *$T$ is continuous if and only if $T$ is bounded.*

**(b)** *If $T$ is continuous at a single point, it is continuous.*

*Proof.* **(a)** For $T=0$ the statement is trivial. Let $T\neq 0$. Then $\|T\|\neq 0$. We assume $T$ to be bounded and consider any $x_0\in\mathfrak{D}(T)$. Let any $\varepsilon>0$ be given. Then, since $T$ is linear, for every $x\in\mathfrak{D}(T)$ such that

$$\|x-x_0\|<\delta \qquad \text{where} \qquad \delta=\frac{\varepsilon}{\|T\|}$$

we obtain

$$\|Tx-Tx_0\|=\|T(x-x_0)\|\leqq\|T\|\,\|x-x_0\|<\|T\|\delta=\varepsilon.$$

Since $x_0\in\mathfrak{D}(T)$ was arbitrary, this shows that $T$ is continuous.

Conversely, assume that $T$ is continuous at an arbitrary $x_0 \in \mathscr{D}(T)$. Then, given any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$(6) \qquad \|Tx - Tx_0\| \leqq \varepsilon \qquad \text{for all } x \in \mathscr{D}(T) \text{ satisfying} \qquad \|x - x_0\| \leqq \delta.$$

We now take any $y \neq 0$ in $\mathscr{D}(T)$ and set

$$x = x_0 + \frac{\delta}{\|y\|} y. \qquad \text{Then} \qquad x - x_0 = \frac{\delta}{\|y\|} y.$$

Hence $\|x - x_0\| = \delta$, so that we may use (6). Since $T$ is linear, we have

$$\|Tx - Tx_0\| = \|T(x - x_0)\| = \left\|T\left(\frac{\delta}{\|y\|} y\right)\right\| = \frac{\delta}{\|y\|} \|Ty\|$$

and (6) implies

$$\frac{\delta}{\|y\|} \|Ty\| \leqq \varepsilon. \qquad \text{Thus} \qquad \|Ty\| \leqq \frac{\varepsilon}{\delta} \|y\|.$$

This can be written $\|Ty\| \leqq c\|y\|$, where $c = \varepsilon/\delta$, and shows that $T$ is bounded.

**(b)** Continuity of $T$ at a point implies boundedness of $T$ by the second part of the proof of $(a)$, which in turn implies continuity of $T$ by $(a)$. ∎

**2.7-10 Corollary (Continuity, null space).** *Let $T$ be a bounded linear operator. Then:*

**(a)** $x_n \longrightarrow x$ *[where $x_n, x \in \mathscr{D}(T)$] implies $Tx_n \longrightarrow Tx$.*

**(b)** *The null space $\mathscr{N}(T)$ is closed.*

*Proof.* **(a)** follows from Theorems 2.7-9($a$) and 1.4-8 or directly from (3) because, as $n \longrightarrow \infty$,

$$\|Tx_n - Tx\| = \|T(x_n - x)\| \leqq \|T\| \|x_n - x\| \longrightarrow 0.$$

**(b)** For every $x \in \overline{\mathscr{N}(T)}$ there is a sequence $(x_n)$ in $\mathscr{N}(T)$ such that $x_n \longrightarrow x$; cf. 1.4-6($a$). Hence $Tx_n \longrightarrow Tx$ by part ($a$) of this Corollary. Also $Tx = 0$ since $Tx_n = 0$, so that $x \in \mathscr{N}(T)$. Since $x \in \overline{\mathscr{N}(T)}$ was arbitrary, $\mathscr{N}(T)$ is closed. ∎

It is worth noting that the range of a bounded linear operator may not be closed. Cf. Prob. 6.

The reader may give the simple proof of another useful formula, namely,

$$\|T_1 T_2\| \leqq \|T_1\| \|T_2\|, \qquad \|T^n\| \leqq \|T\|^n \qquad (n \in \mathbf{N}) \tag{7}$$

valid for bounded linear operators $T_2\colon X \longrightarrow Y$, $T_1\colon Y \longrightarrow Z$ and $T\colon X \longrightarrow X$, where $X$, $Y$, $Z$ are normed spaces.

Two operators $T_1$ and $T_2$ are defined to be **equal**, written

$$T_1 = T_2,$$

if they have the same domain $\mathscr{D}(T_1) = \mathscr{D}(T_2)$ and if $T_1 x = T_2 x$ for all $x \in \mathscr{D}(T_1) = \mathscr{D}(T_2)$.

The **restriction** of an operator $T\colon \mathscr{D}(T) \longrightarrow Y$ to a subset $B \subset \mathscr{D}(T)$ is denoted by

$$T|_B$$

and is the operator defined by

$$T|_B\colon B \longrightarrow Y, \qquad T|_B x = Tx \text{ for all } x \in B.$$

An **extension** of $T$ to a set $M \supset \mathfrak{D}(T)$ is an operator

$$\tilde{T}: M \longrightarrow Y \qquad \text{such that} \qquad \tilde{T}|_{\mathfrak{D}(T)} = T,$$

that is, $\tilde{T}x = Tx$ for all $x \in \mathfrak{D}(T)$. [Hence $T$ is the restriction of $\tilde{T}$ to $\mathfrak{D}(T)$.]

**2.7-11 Theorem (Bounded linear extension).** *Let*

$$T: \mathfrak{D}(T) \longrightarrow Y$$

*be a bounded linear operator, where $\mathfrak{D}(T)$ lies in a normed space $X$ and $Y$ is a Banach space. Then $T$ has an extension*

$$\tilde{T}: \overline{\mathfrak{D}(T)} \longrightarrow Y$$

*where $\tilde{T}$ is a bounded linear operator of norm*

$$\|\tilde{T}\| = \|T\|.$$

*Proof.* We consider any $x \in \overline{\mathcal{D}(T)}$. By Theorem 1.4-6(*a*) there is a sequence $(x_n)$ in $\mathcal{D}(T)$ such that $x_n \longrightarrow x$. Since $T$ is linear and bounded, we have

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leqq \|T\| \|x_n - x_m\|.$$

This shows that $(Tx_n)$ is Cauchy because $(x_n)$ converges. By assumption, $Y$ is complete, so that $(Tx_n)$ converges, say,

$$Tx_n \longrightarrow y \in Y.$$

We define $\tilde{T}$ by

$$\tilde{T}x = y.$$

We show that this definition is independent of the particular choice of a sequence in $\mathcal{D}(T)$ converging to $x$. Suppose that $x_n \longrightarrow x$ and $z_n \longrightarrow x$. Then $v_m \longrightarrow x$, where $(v_m)$ is the sequence

$$(x_1, z_1, x_2, z_2, \cdots).$$

Hence $(Tv_m)$ converges by 2.7-10(*a*), and the two subsequences $(Tx_n)$ and $(Tz_n)$ of $(Tv_m)$ must have the same limit. This proves that $\tilde{T}$ is uniquely defined at every $x \in \overline{\mathfrak{D}(T)}$.

Clearly, $\tilde{T}$ is linear and $\tilde{T}x = Tx$ for every $x \in \mathfrak{D}(T)$, so that $\tilde{T}$ is an extension of $T$. We now use

$$\|Tx_n\| \leqq \|T\| \|x_n\|$$

and let $n \longrightarrow \infty$. Then $Tx_n \longrightarrow y = \tilde{T}x$. Since $x \longmapsto \|x\|$ defines a continuous mapping (cf. Sec. 2.2), we thus obtain

$$\|\tilde{T}x\| \leqq \|T\| \|x\|.$$

Hence $\tilde{T}$ is bounded and $\|\tilde{T}\| \leqq \|T\|$. Of course, $\|\tilde{T}\| \geqq \|T\|$ because the norm, being defined by a supremum, cannot decrease in an extension. Together we have $\|\tilde{T}\| = \|T\|$. ∎

پایان

با تشکر