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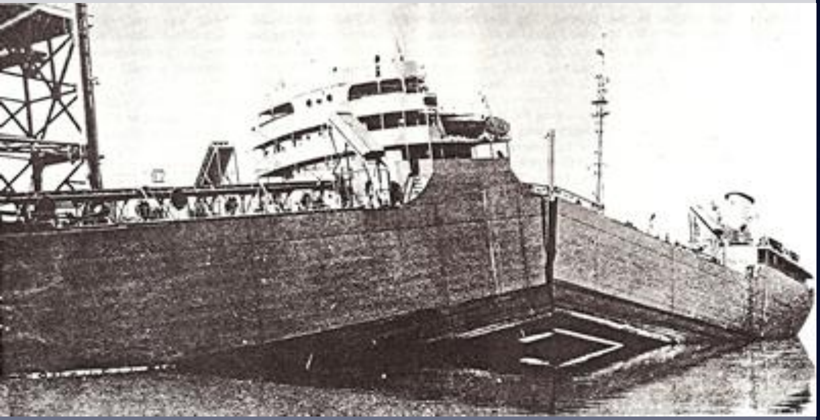
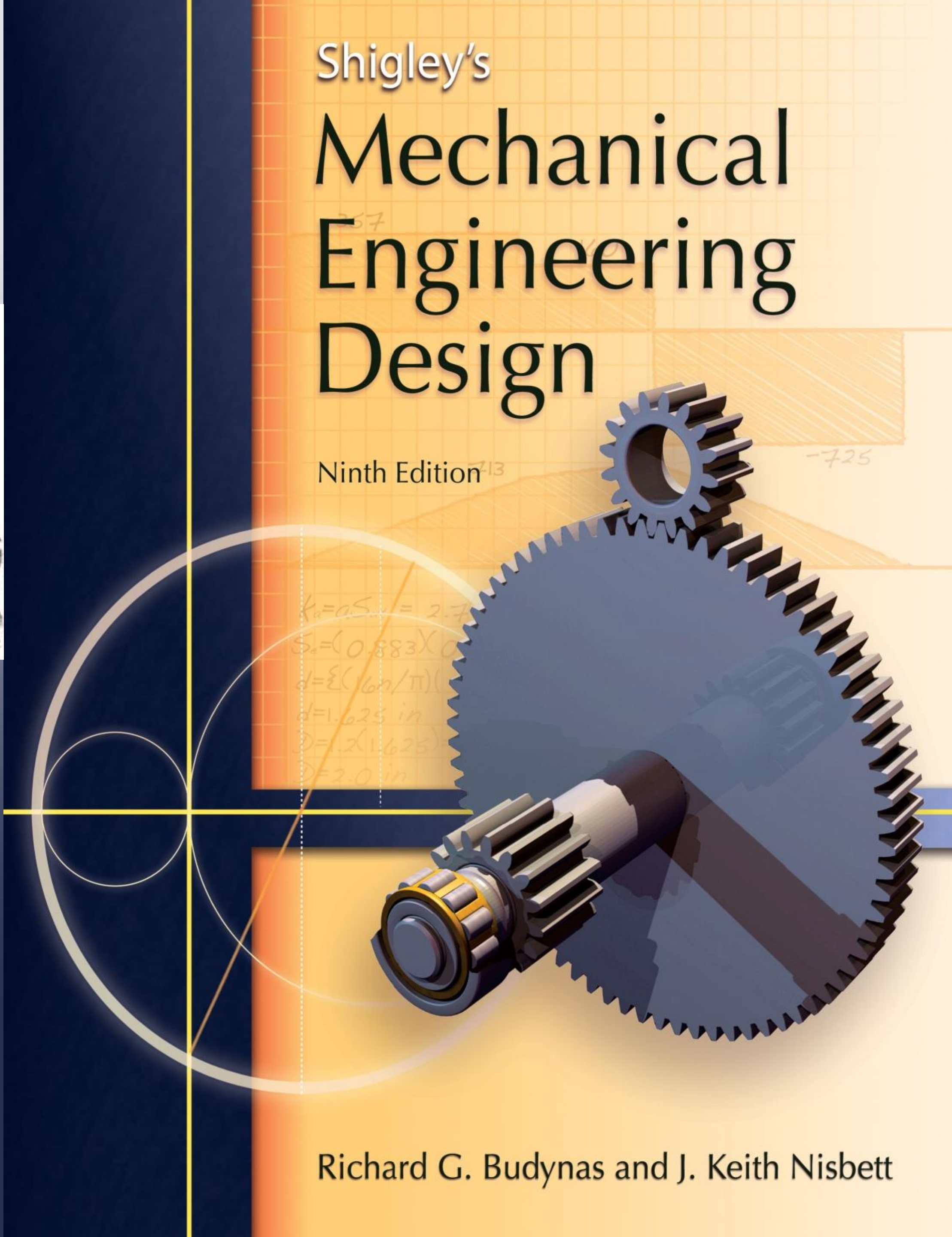
Lecture Slides



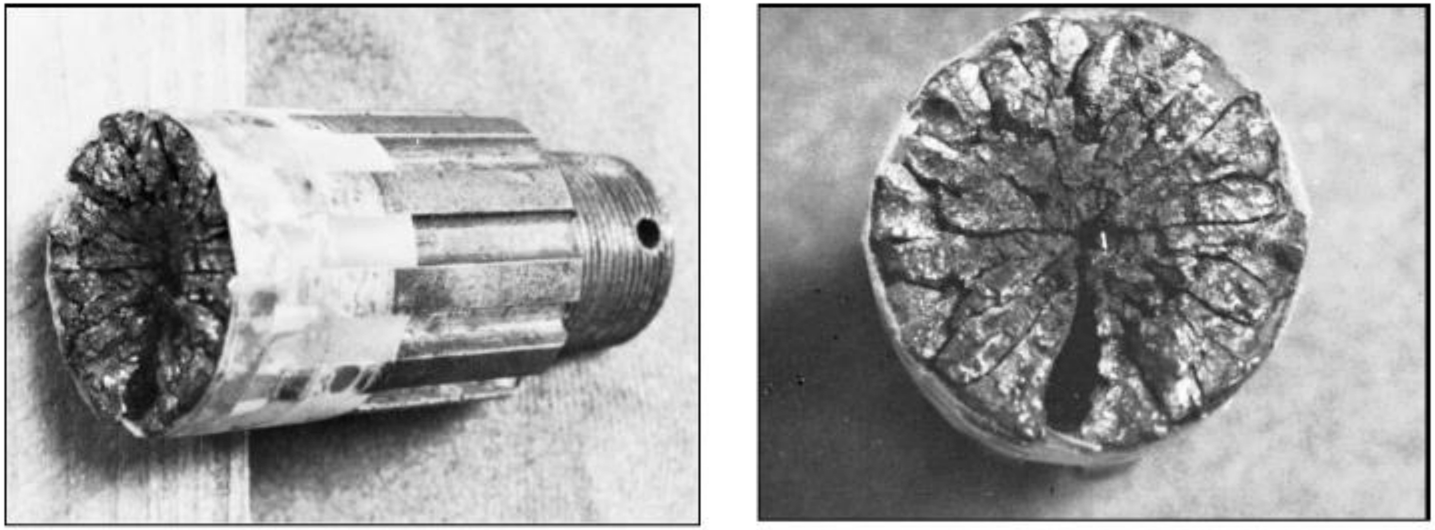
Chapter 5

Failures Resulting from Static Loading



Chapter Outline

Failure Examples



(a) (b)

Fig. 5–1

- Failure of truck driveshaft spline due to corrosion fatigue

Failure Examples

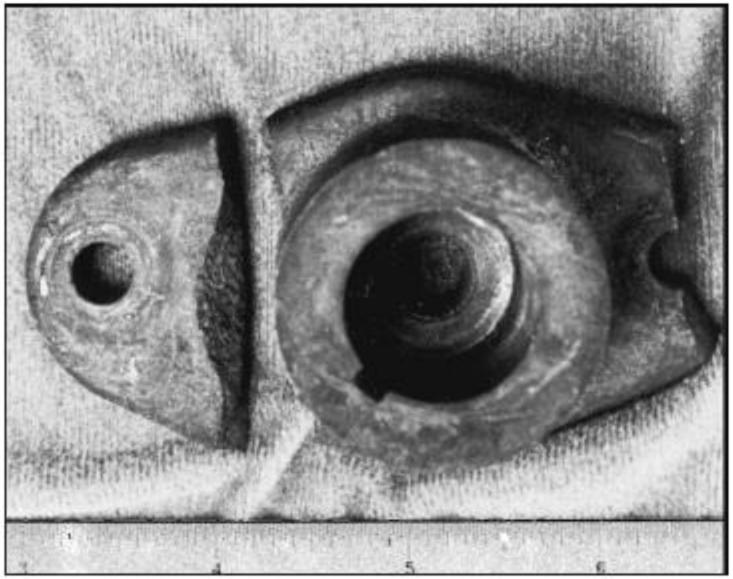


Fig. 5–2

- Impact failure of a lawn-mower blade driver hub.
- The blade impacted a surveying pipe marker.

Failure Examples

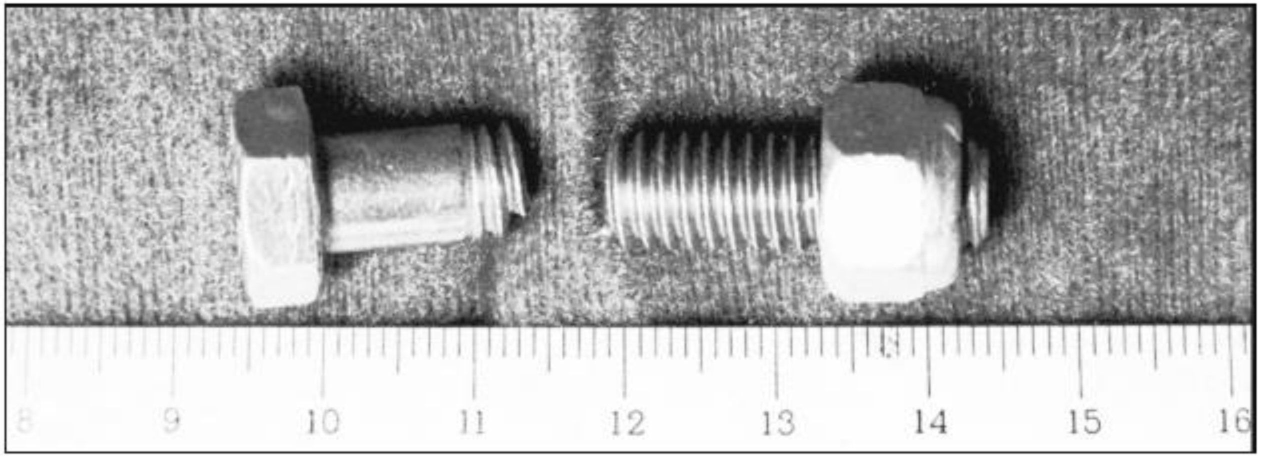


Fig. 5–3

- Failure of an overhead-pulley retaining bolt on a weightlifting machine.
- A manufacturing error caused a gap that forced the bolt to take the entire moment load.

Failure Examples

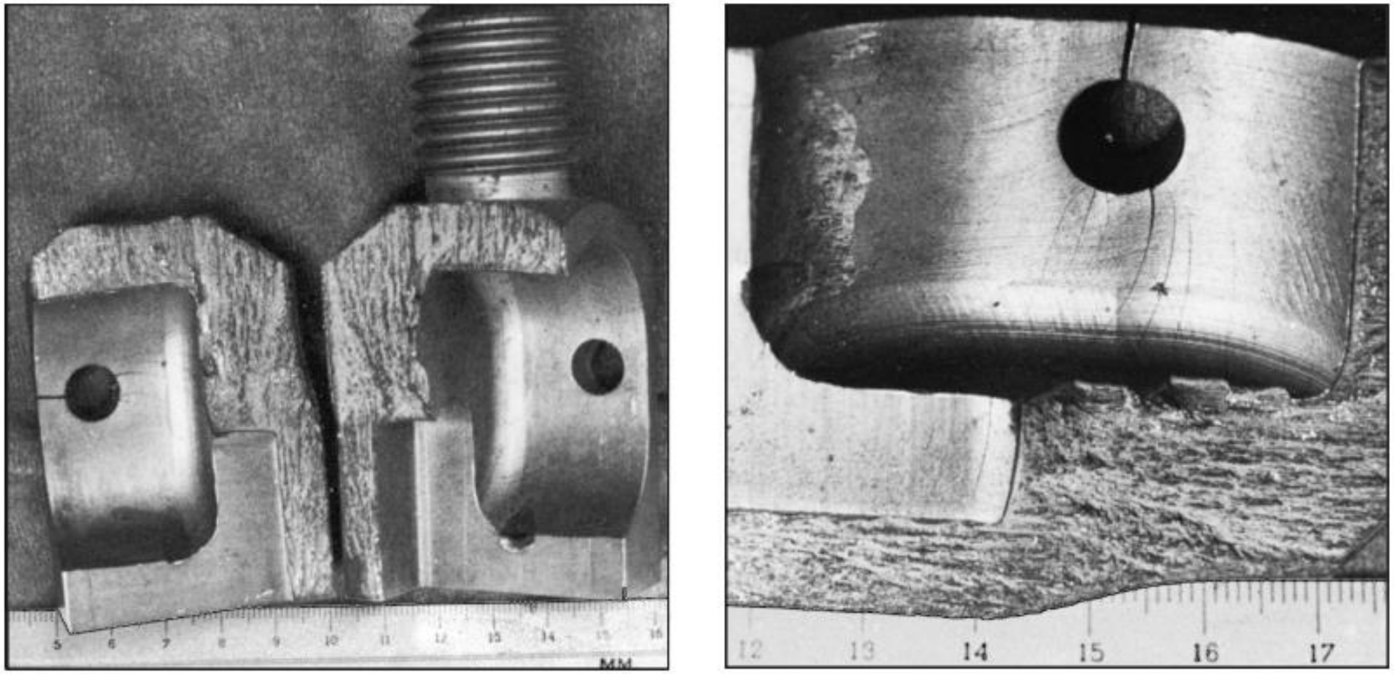


Fig. 5–4

(a) (b)

- Chain test fixture that failed in one cycle.
- To alleviate complaints of excessive wear, the manufacturer decided to case-harden the material
- (a) Two halves showing brittle fracture initiated by stress concentration
- (b) Enlarged view showing cracks induced by stress concentration at the support-pin holes

Failure Examples

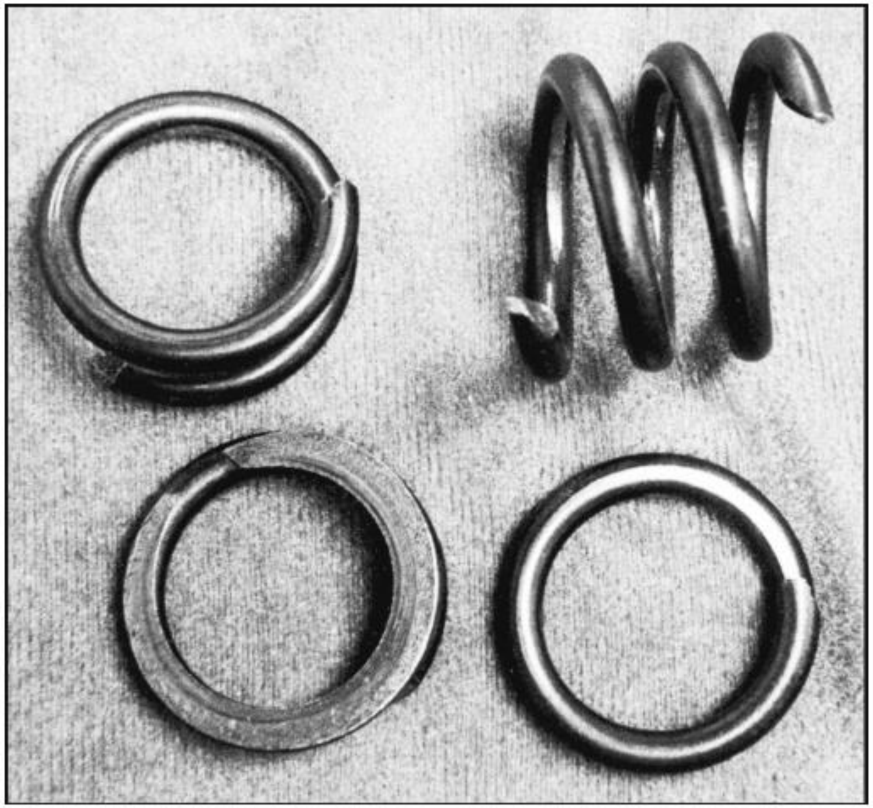


Fig. 5–5

- Valve-spring failure caused by spring surge in an oversped engine.
- The fractures exhibit the classic 45 degree shear failure

Static Strength

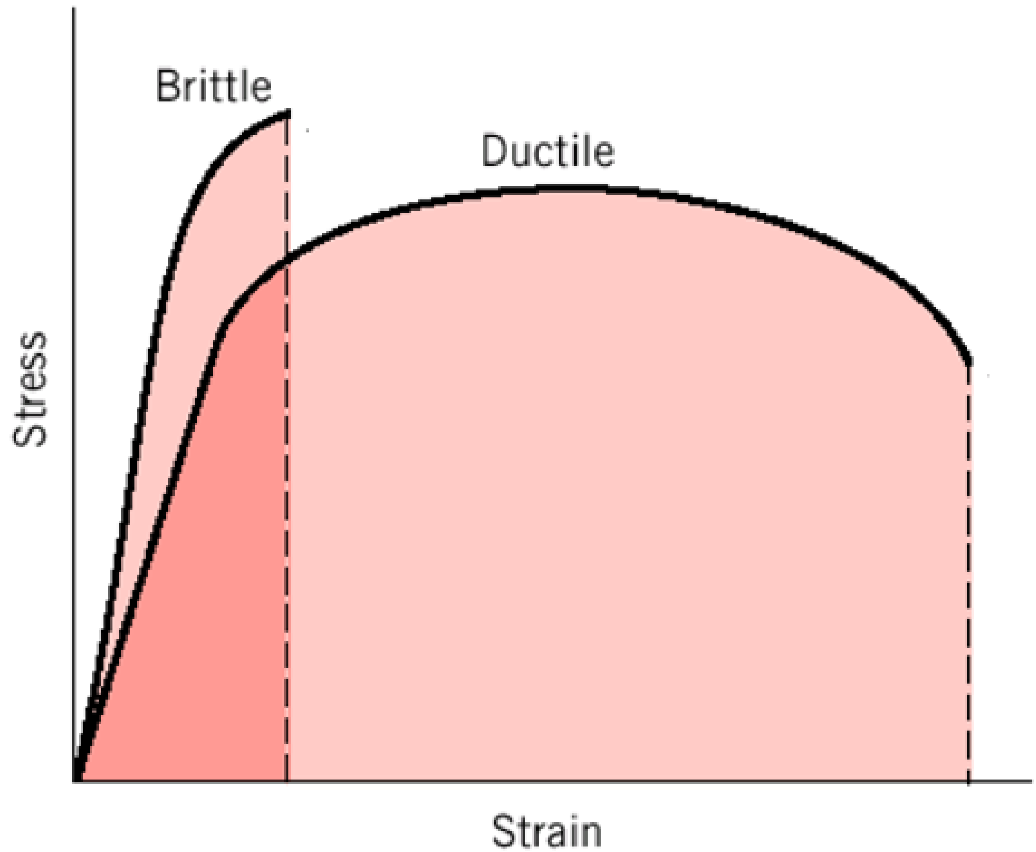
- Failure of the part would endanger human life, or the part is made in extremely large quantities; consequently, an elaborate testing program is justified during design.
- The part is made in large enough quantities that a moderate series of tests is feasible.
- The part is made in such small quantities that testing is not justified at all; or the design must be completed so rapidly that there is not enough time for testing.
- Experimental test data is better, but generally only warranted for large quantities or when failure is very costly (in time, expense, or life)
- The part has already been designed, manufactured, and tested and found to be unsatisfactory. Analysis is required to understand why the part is unsatisfactory and what to do to improve it.

Ductility and Percent Elongation

- *Ductility* is the degree to which a material will deform before ultimate fracture.
- Percent elongation is used as a measure of ductility.
- Ductile Materials have % $\varepsilon \geq 5\%$
- Brittle Materials have % $\varepsilon < 5\%$
- For machine members subject to repeated or shock or impact loads, materials with % $\varepsilon > 12\%$ are recommended.

Ductile materials - extensive plastic deformation and energy absorption (toughness) before fracture

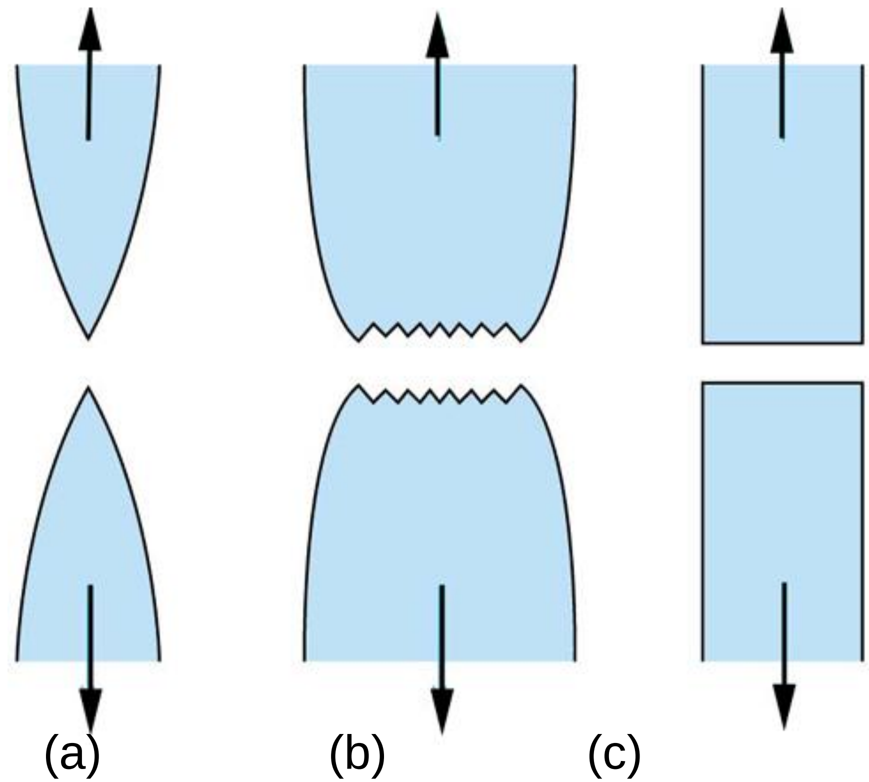
Brittle materials - little plastic deformation and low energy absorption before failure



DUCTILE VS BRITTLE FAILURE

- Classification:

(*a*) Highly ductile fracture in which the specimen necks down to a point. (*b*) Moderately ductile fracture after some necking. (*c*) Brittle fracture without any plastic deformation.



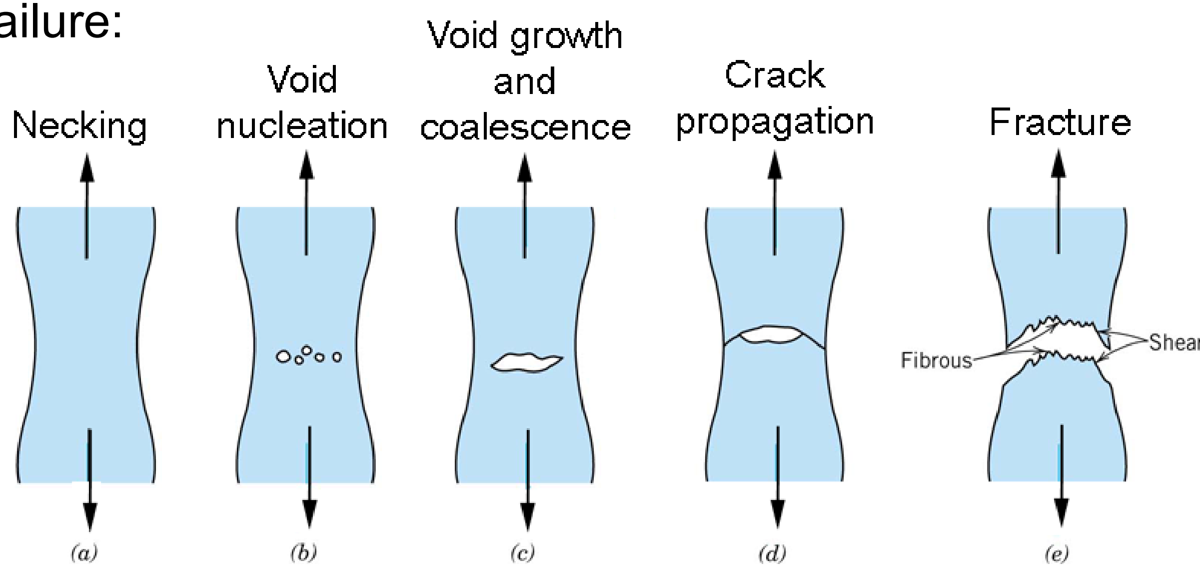
- Ductile fracture is desirable!

Ductile: warning before fracture

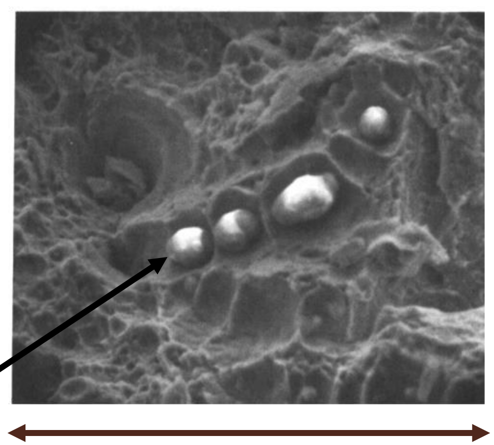
Brittle: No warning

DUCTILE FAILURE

- Evolution to failure:



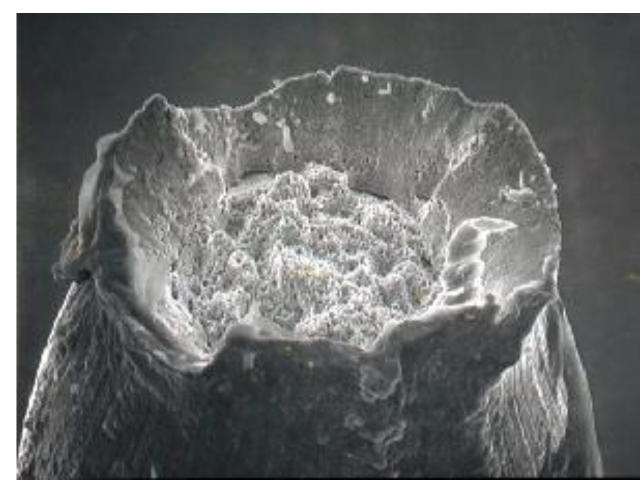
- Resulting fracture surfaces (steel)



50 µm

particles serve as void nucleation sites.

"cup and cone" fracture

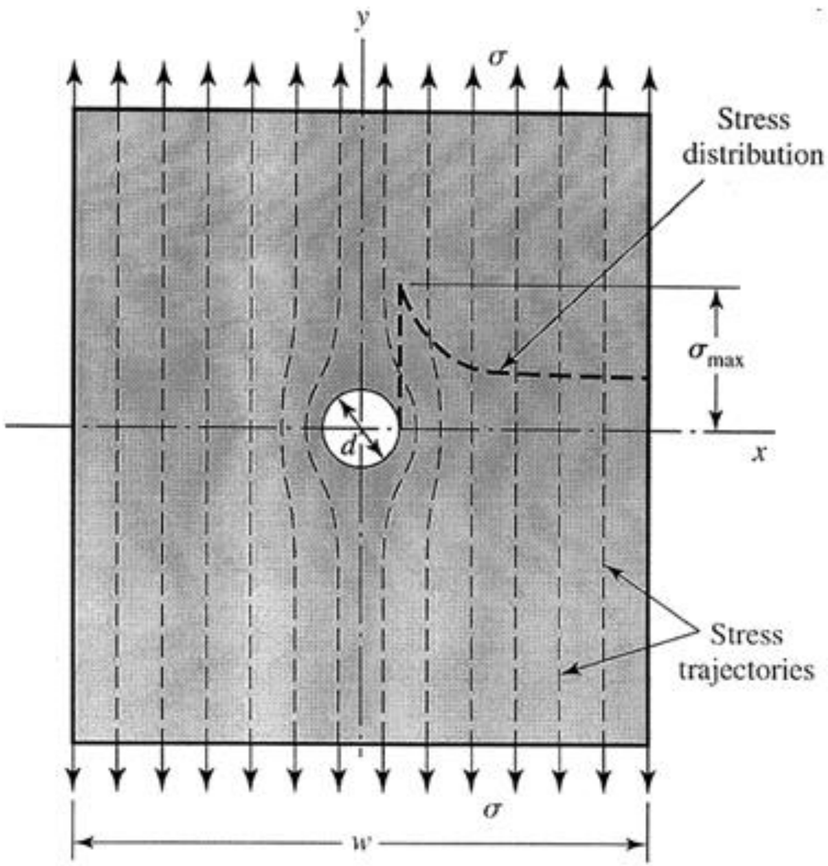


1 µm = 1 X 10^{-6} m = 0.001 mm

Stress Concentration

- Localized increase of stress near discontinuities
- K_t is Theoretical (Geometric) Stress Concentration Factor

$$K_t = \frac{\sigma_{\max}}{\sigma_0} \qquad K_{ts} = \frac{\tau_{\max}}{\tau_0} \qquad (3\text{–}48)$$



Theoretical Stress Concentration Factor

- Graphs available for standard configurations
- See Appendix A–15 and A–16 for common examples
- Many more in *Peterson's Stress-Concentration Factors*
- Note the trend for higher K_t at sharper discontinuity radius, and at greater disruption

Figure A–15–1

Bar in tension or simple compression with a transverse hole. $\sigma_0 = F/A$, where $A = (w - d)t$ and t is the thickness.

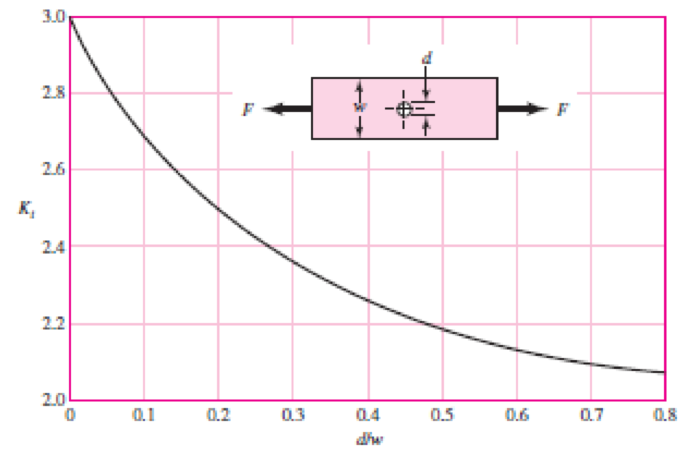
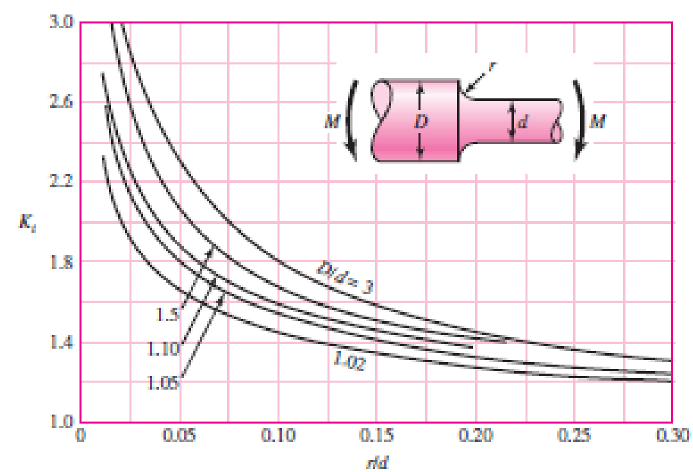


Figure A–15–9

Round shaft with shoulder fillet in bending. $\sigma_0 = Mc/I$, where $c = d/2$ and $I = \pi d^4/64$.



Stress Concentration for Static and Ductile Conditions

- With static loads and ductile materials
  - Highest stressed fibers yield (cold work)
  - Load is shared with next fibers
  - Cold working is localized
  - Overall part does not see damage unless ultimate strength is exceeded
  - Stress concentration effect is commonly ignored for static loads on ductile materials
- Stress concentration must be included for dynamic loading (See Ch. 6)
- Stress concentration must be included for brittle materials, since localized yielding may reach brittle failure rather than cold-working and sharing the load.

Need for Static Failure Theories

Failure theories are used to predict if failure would occur under any given state of stress

The generally accepted theories are:

- **Ductile materials** (yield criteria)
  - Maximum shear stress (MSS),
  - Distortion energy (DE),
  - Ductile Coulomb-Mohr (DCM),

- **Brittle materials** (fracture criteria)
  - Maximum normal stress (MNS),
  - Brittle Coulomb-Mohr (BCM),
  - Modified Mohr (MM),

Maximum Shear Stress Theory (MSS)

- Theory: Yielding begins when the *maximum shear stress* in a stress element exceeds the maximum shear stress in a tension test specimen of the same material when that specimen begins to yield.
- For a tension test specimen, the maximum shear stress is $\sigma_1/2$.
- At yielding, when $\sigma_1 = S_y$, the maximum shear stress is $S_y/2$.
- Could restate the theory as follows:
  - Theory: Yielding begins when the *maximum shear stress* in a stress element exceeds $S_y/2$.

Maximum Shear Stress Theory (MSS)

- For any stress element, use Mohr's circle to find the maximum shear stress. Compare the maximum shear stress to $S_y/2$.
- Ordering the principal stresses such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$,

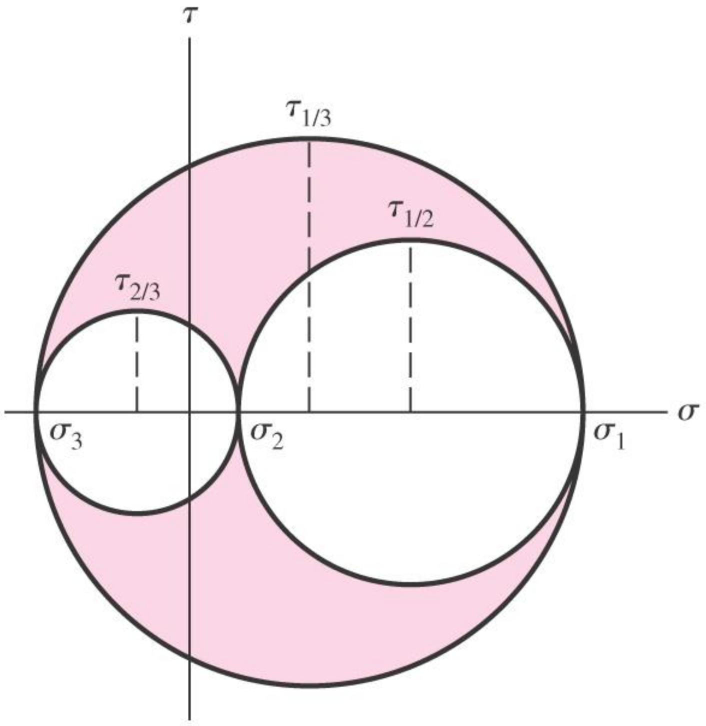
$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{S_y}{2} \qquad \text{or} \qquad \sigma_1 - \sigma_3 \geq S_y \qquad (5\text{–}1)$$

- Incorporating a design factor n

$$\tau_{max} = \frac{S_y}{2n} \qquad \text{or} \qquad \sigma_1 - \sigma_3 = \frac{S_y}{n}$$

- Or solving for factor of safety

$$n = \frac{S_y / 2}{\tau_{max}}$$



Maximum Shear Stress Theory (MSS)

- To compare to experimental data, express $\tau_{\max}$ in terms of principal stresses and plot.
- To simplify, consider a plane stress state (*one of the principal stress is zero*)
- Let σ_A and σ_B represent the two non-zero principal stresses, then order them with the zero principal stress such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$
- Assuming $\sigma_A \geq \sigma_B$ there are three cases to consider
  - Case 1: $\sigma_A \geq \sigma_B \geq 0$
  - Case 2: $\sigma_A \geq 0 \geq \sigma_B$
  - Case 3: $0 \geq \sigma_A \geq \sigma_B$

Maximum Shear Stress Theory (MSS)

- *Case 1*: $\sigma_A \geq \sigma_B \geq 0$
  - For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$
  - Eq. (5–1) reduces to $\sigma_A \geq S_y$
  - $\boldsymbol{\sigma_A = S_y/n}$.
- *Case 2*: $\sigma_A \geq 0 \geq \sigma_B$
  - For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = \sigma_B$
  - Eq. (5–1) reduces to $\sigma_A - \sigma_B \geq S_y$
  - $\boldsymbol{(\sigma_A - \sigma_B) = S_y/n}$.
- Case 3: $0 \geq \sigma_A \geq \sigma_B$
  - For this case, $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$
  - Eq. (5–1) reduces to $\sigma_B \leq -S_y$
  - $\boldsymbol{\sigma_B = -S_y/n}$.

Maximum Shear Stress Theory (MSS)

- Plot three cases on principal stress axes
- Case 1: $\sigma_A \geq \sigma_B \geq 0$
  - $\sigma_A \geq S_y$
- Case 2: $\sigma_A \geq 0 \geq \sigma_B$
  - $\sigma_A - \sigma_B \geq S_y$
- Case 3: $0 \geq \sigma_A \geq \sigma_B$
  - $\sigma_B \leq -S_y$
- Other lines are symmetric cases
- Inside envelope is predicted safe zone

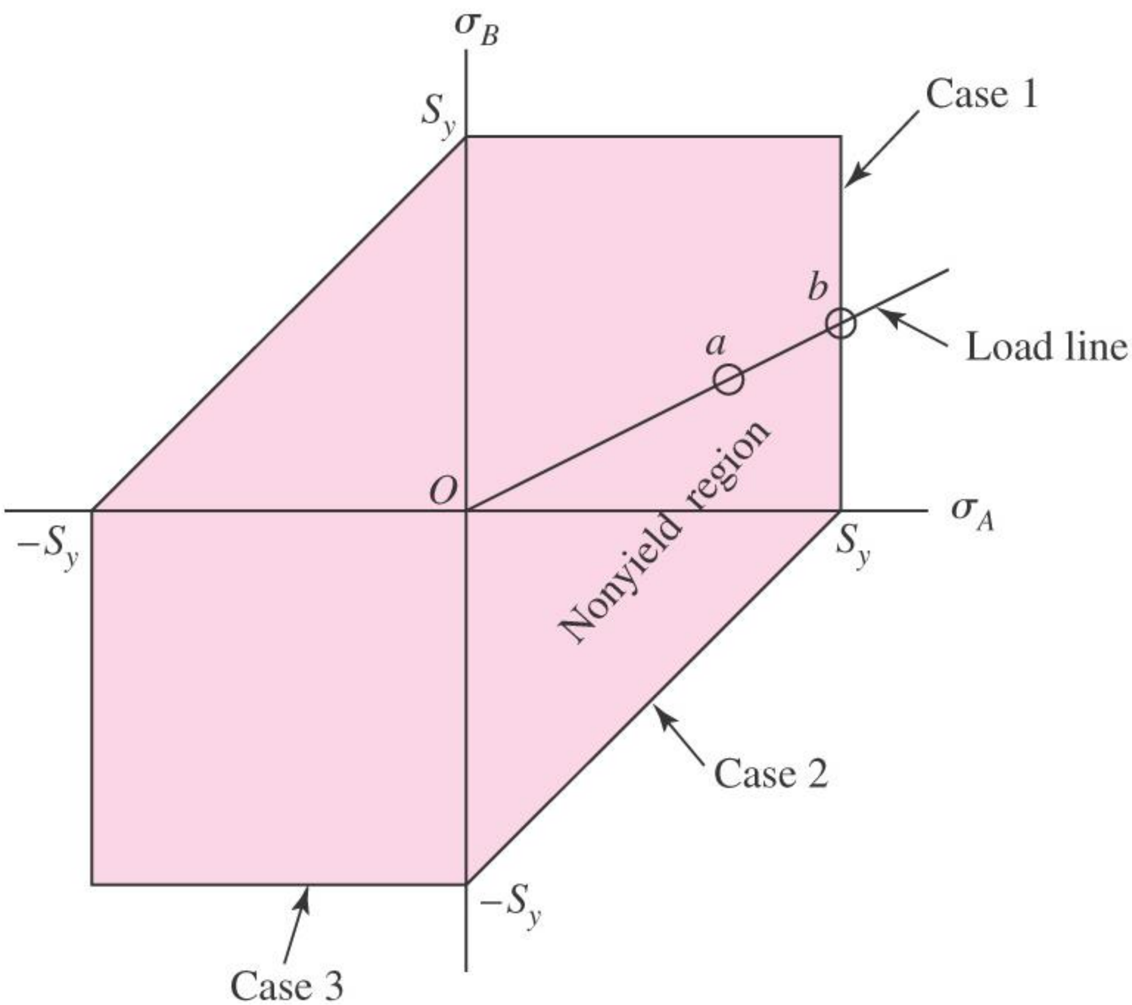
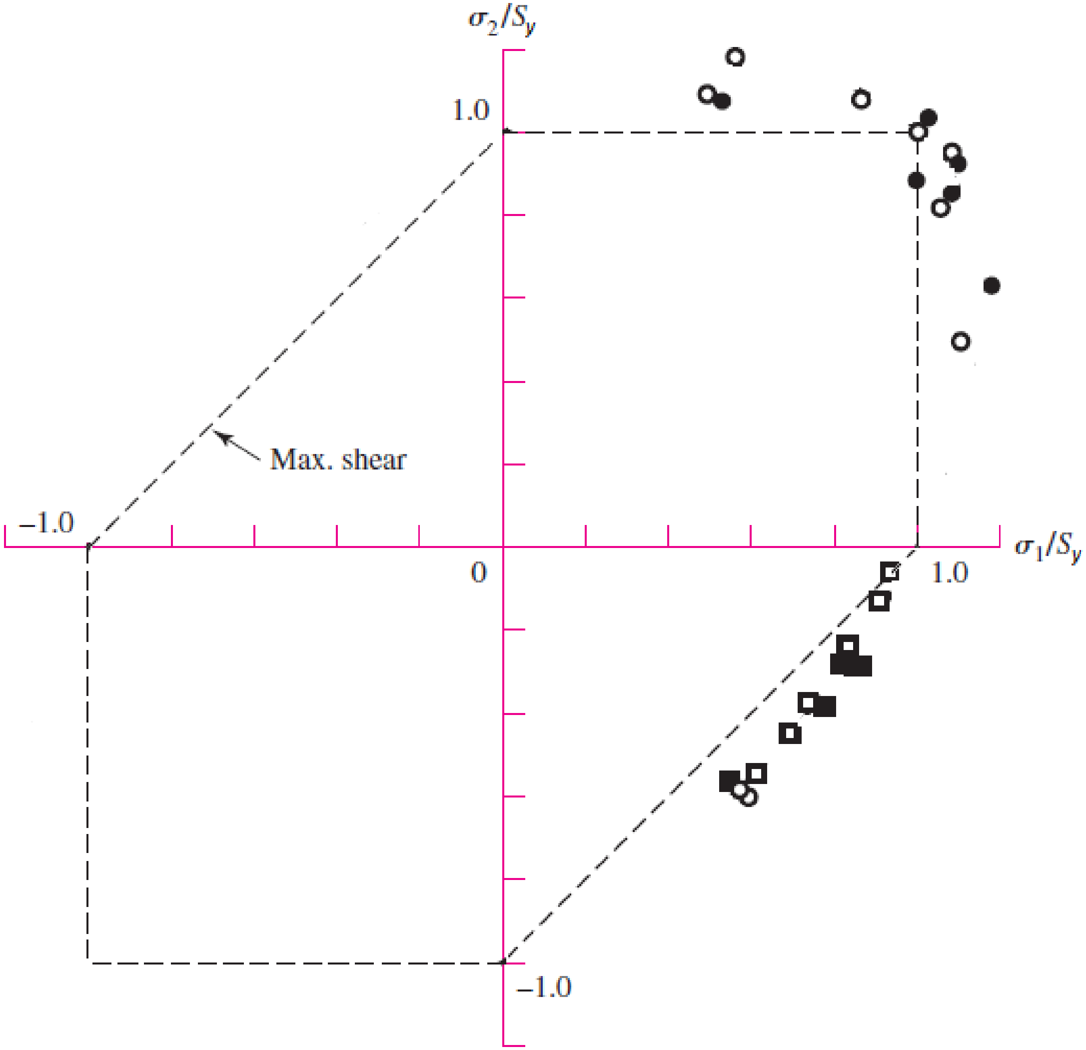


Fig. 5–7

Maximum Shear Stress Theory (MSS)

- Comparison to experimental data
- Conservative in all quadrants
- Commonly used for design situations



Distortion Energy (DE) Failure Theory

- Also known as:
  - Octahedral Shear Stress
  - Shear Energy
  - Von Mises
  - Von Mises – Hencky

Distortion Energy (DE) Failure Theory

- Originated from observation that ductile materials stressed hydrostatically (equal principal stresses) exhibited yield strengths greatly in excess of expected values.
- Theorizes that if strain energy is divided into hydrostatic volume changing energy and angular distortion energy, the yielding is primarily affected by the distortion energy.

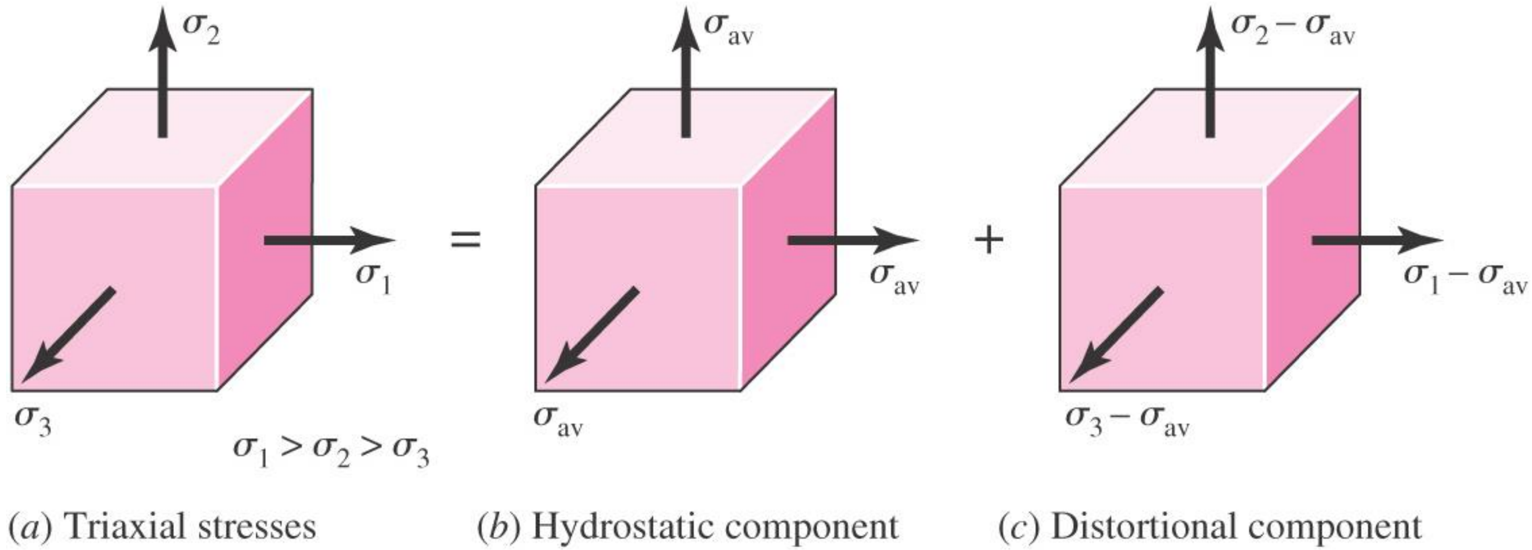


Fig. 5–8

Distortion Energy (DE) Failure Theory

- Theory: Yielding occurs when the *distortion strain energy* per unit volume reaches the distortion strain energy per unit volume for yield in simple tension or compression of the same material.

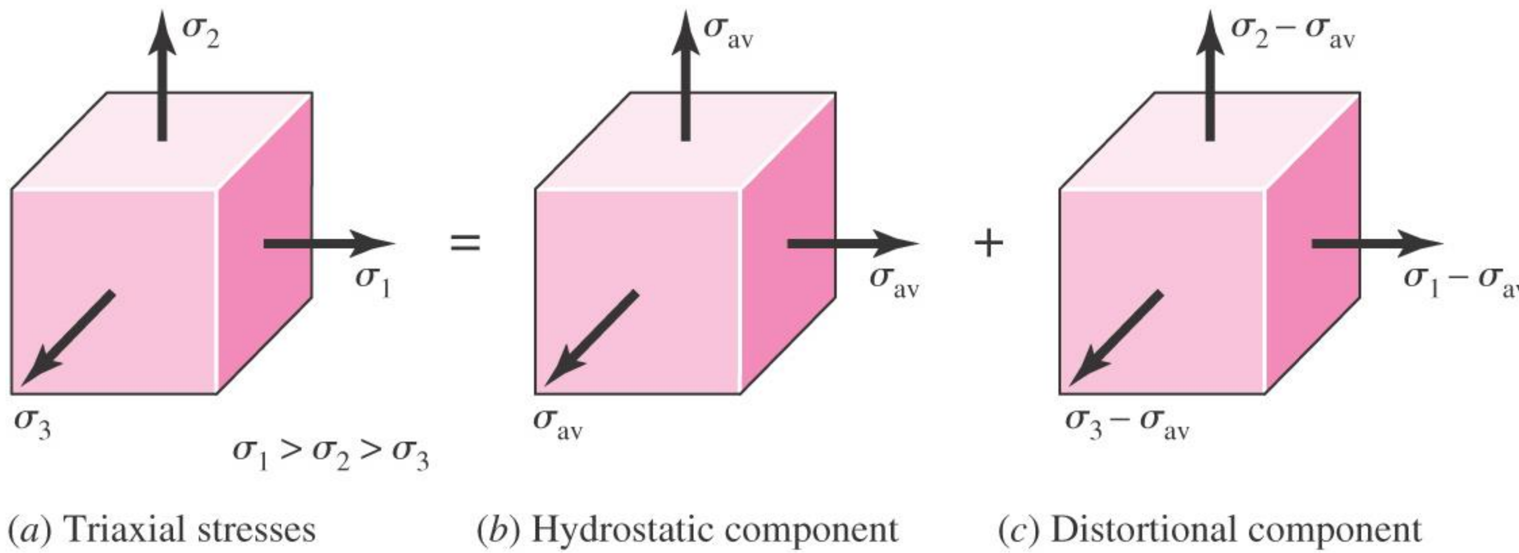


Fig. 5–8

Deriving the Distortion Energy

- Hydrostatic stress is average of principal stresses

$$\sigma_{av} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \qquad (a)$$

- Strain energy per unit volume, $u = \frac{1}{2}[\epsilon_1\sigma_1 + \epsilon_2\sigma_2 + \epsilon_3\sigma_3]$
- Substituting Eq. (3–19) for principal strains into strain energy equation,

$$\begin{aligned} \epsilon_x &= \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \\ \epsilon_y &= \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] \\ \epsilon_z &= \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \end{aligned} \qquad (3\text{–}19)$$

$$u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \qquad (b)$$

Deriving the Distortion Energy

$$u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \qquad (b)$$

- Strain energy for producing only volume change is obtained by substituting σ_{av} for σ_1, σ_2, and σ_3

$$u_v = \frac{3\sigma_{av}^2}{2E}(1 - 2\nu) \qquad (c)$$

- Substituting σ_{av} from Eq. (a),

$$u_v = \frac{1 - 2\nu}{6E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_3\sigma_1\right) \qquad (5\text{–}7)$$

- Obtain distortion energy by subtracting volume changing energy, Eq. (5–7), from total strain energy, Eq. (b)

$$u_d = u - u_v = \frac{1 + \nu}{3E}\left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right] \qquad (5\text{–}8)$$

Deriving the Distortion Energy

$$u_d = u - u_v = \frac{1+\nu}{3E}\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right] \quad (5\text{–}8)$$

- Tension test specimen at yield has $\sigma_1 = S_y$ and $\sigma_2 = \sigma_3 = 0$
- Applying to Eq. (5–8), distortion energy for tension test specimen is

$$u_d = \frac{1+\nu}{3E} S_y^2 \quad (5\text{–}9)$$

- DE theory predicts failure when distortion energy, Eq. (5–8), exceeds distortion energy of tension test specimen, Eq. (5–9)

$$\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]^{1/2} \geq S_y \quad (5\text{–}10)$$

Von Mises Stress

$$\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]^{1/2} \geq S_y \qquad (5\text{–}10)$$

- Left hand side is defined as *von Mises stress*

$$\sigma' = \left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]^{1/2} \qquad (5\text{–}12)$$

- For plane stress, simplifies to

$$\sigma' = \left(\sigma_A^2 - \sigma_A\sigma_B + \sigma_B^2\right)^{1/2} \qquad (5\text{–}13)$$

- In terms of *xyz* components, in three dimensions

$$\sigma' = \frac{1}{\sqrt{2}}\left[(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2\right)\right]^{1/2} \qquad (5\text{–}14)$$

- In terms of *xyz* components, for plane stress

$$\sigma' = \left(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2\right)^{1/2} \qquad (5\text{–}15)$$

Distortion Energy Theory With Von Mises Stress

- Von Mises Stress can be thought of as a single, equivalent, or effective stress for the entire general state of stress in a stress element.
- Distortion Energy failure theory simply compares von Mises stress to yield strength.

$$\sigma' \geq S_y \tag{5–11}$$

- Introducing a design factor,

$$\sigma' = \frac{S_y}{n} \tag{5–19}$$

- Expressing as factor of safety,

$$n = \frac{S_y}{\sigma'}$$

Failure Theory in Terms of von Mises Stress

- Equation is identical to Eq. (5–10) from Distortion Energy approach
- Identical conclusion for:
  - Distortion Energy
  - Octahedral Shear Stress
  - Shear Energy
  - Von Mises
  - Von Mises – Hencky

$$n = \frac{S_y}{\sigma'}$$

DE Theory Compared to Experimental Data

- Plot von Mises stress on principal stress axes to compare to experimental data (and to other failure theories)
- DE curve is *typical* of data
- Note that *typical* equates to a 50% reliability from a design perspective
- Commonly used for analysis situations
- MSS theory useful for design situations where higher reliability is desired

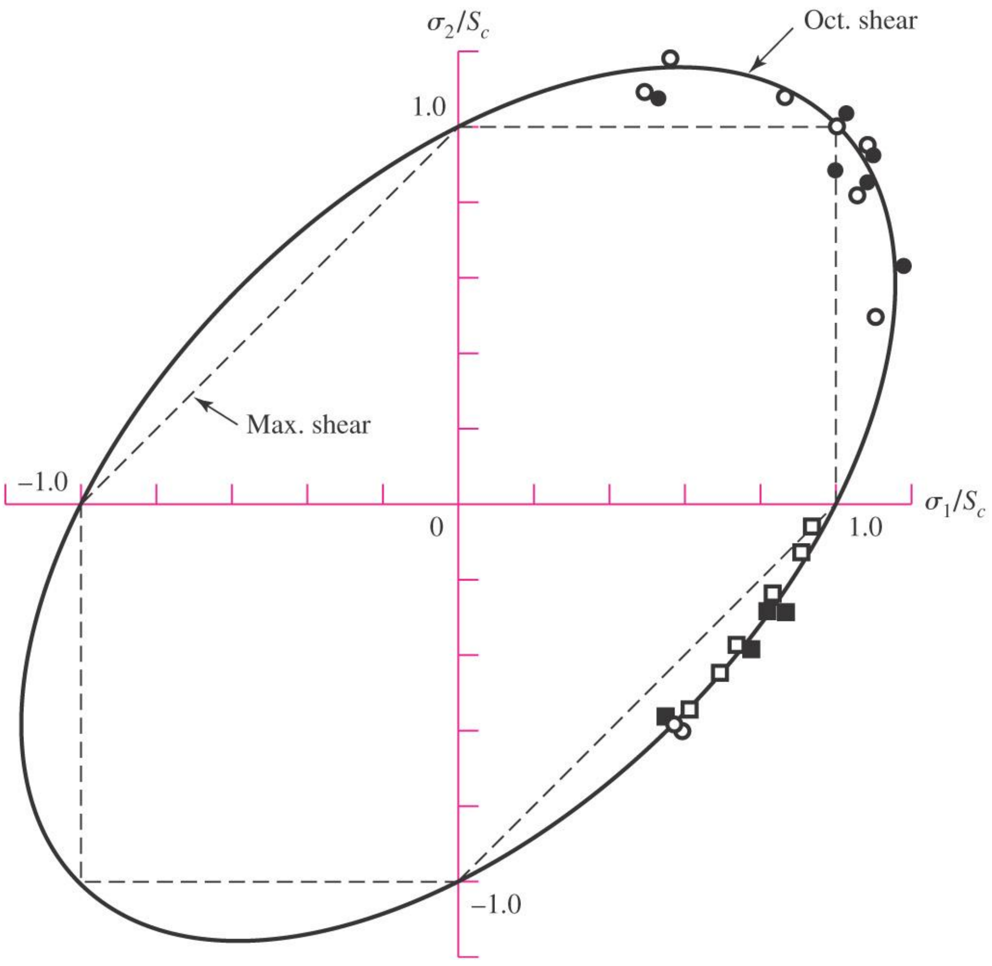


Fig. 5–15

Shear Strength Predictions

- For pure shear loading, Mohr's circle shows that $\sigma_A = -\sigma_B = \tau$
- Plotting this equation on principal stress axes gives load line for pure shear case
- Intersection of pure shear load line with failure curve indicates shear strength has been reached
- Each failure theory predicts shear strength to be some fraction of normal strength

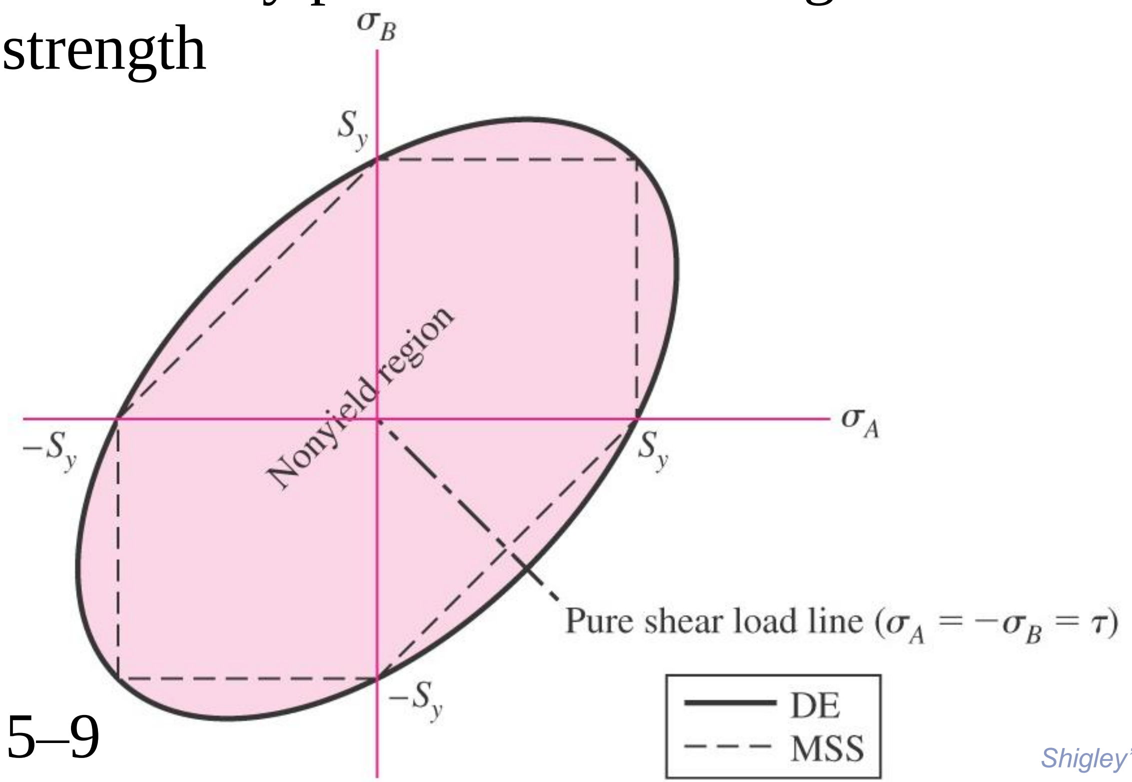


Fig. 5–9

Example 5-1

EXAMPLE 5–1 A hot-rolled steel has a yield strength of $S_{yt} = S_{yc} = 700$ MPa and a true strain at fracture of $\varepsilon_f = 0.55$. Estimate the factor of safety for the following principal stress states:

(*a*) 490, 490, 0 MPa.
(*b*) 210, 490, 0 MPa.
(*c*) 0, 490, −210 MPa.
(*d*) 0, −210, −490 MPa.
(*e*) 210, 210, 210 MPa.

Solution Since $\varepsilon_f > 0.05$ and S_{yc} and S_{yt} are equal, the material is ductile and the distortion-energy (DE) theory applies. The maximum-shear-stress (MSS) theory will also be applied and compared to the DE results. Note that cases *a* to *d* are plane stress states.

(*a*) The ordered principal stresses are $\sigma_A = \sigma_1 = 490$, $\sigma_B = \sigma_2 = 490$, $\sigma_3 = 0$ MPa.

DE From Eq. (5–13), $\sigma' = \left(\sigma_A^2 - \sigma_A\sigma_B + \sigma_B^2\right)^{1/2}$ (5–13)

$$\sigma' = [490^2 - 490(490) + 490^2]^{1/2} = 490 \text{ MPa}$$

Answer
$$n = \frac{S_y}{\sigma'} = \frac{700}{490} = 1.43$$

MSS Case 1, using Eq. (5–4) with a factor of safety,

Answer
$$n = \frac{S_y}{\sigma_A} = \frac{700}{490} = 1.43$$

(*b*) The ordered principal stresses are $\sigma_A = \sigma_1 = 490$, $\sigma_B = \sigma_2 = 210$, $\sigma_3 = 0$ MPa.

DE
$$\sigma' = [490^2 - 490(210) + 210^2]^{1/2} = 426 \text{ MPa}$$

Answer
$$n = \frac{S_y}{\sigma'} = \frac{700}{426} = 1.64$$

Example 5-1

MSS Case 1, using Eq. (5–4),

Answer $$n = \frac{S_y}{\sigma_A} = \frac{700}{490} = 1.43$$

(*c*) The ordered principal stresses are $\sigma_A = \sigma_1 = 490$, $\sigma_2 = 0$, $\sigma_B = \sigma_3 = -210$ MPa.

DE $$\sigma' = [490^2 - 490(-210) + (-210)^2]^{1/2} = 622 \text{ MPa}$$

Answer $$n = \frac{S_y}{\sigma'} = \frac{700}{622} = 1.13$$

MSS Case 2, using Eq. (5–5),

Answer $$n = \frac{S_y}{\sigma_A - \sigma_B} = \frac{700}{490 - (-210)} = 1.00$$

(*d*) The ordered principal stresses are $\sigma_1 = 0$, $\sigma_A = \sigma_2 = -210$, $\sigma_B = \sigma_3 = -490$ MPa.

DE $$\sigma' = [(-490)^2 - (-490)(-210) + (-210)^2]^{1/2} = 426 \text{ MPa}$$

Answer $$n = \frac{S_y}{\sigma'} = \frac{700}{426} = 1.64$$

MSS Case 3, using Eq. (5–6),

Answer $$n = -\frac{S_y}{\sigma_B} = -\frac{700}{-490} = 1.43$$

Example 5-1

(*e*) The ordered principal stresses are $\sigma_1 = 210$, $\sigma_2 = 210$, $\sigma_3 = 210$ MPa

DE From Eq. (5–12),

$$\sigma' = \left[\frac{(210-210)^2 + (210-210)^2 + (210-210)^2}{2}\right]^{1/2} = 0 \text{ MPa}$$

Answer

$$n = \frac{S_y}{\sigma'} = \frac{700}{0} \rightarrow \infty$$

MSS From Eq. (5–3),

Answer

$$n = \frac{S_y}{\sigma_1 - \sigma_3} = \frac{700}{210-210} \rightarrow \infty$$

A tabular summary of the factors of safety is included for comparisons.

	(*a*)	(*b*)	(*c*)	(*d*)	(*e*)
DE	1.43	1.64	1.13	1.64	∞
MSS	1.43	1.43	1.00	1.43	∞

Example 5-1

A tabular summary of the factors of safety is included for comparisons.

	(a)	(b)	(c)	(d)	(e)
DE	1.43	1.64	1.13	1.64	∞
MSS	1.43	1.43	1.00	1.43	∞

Since the MSS theory is on or within the boundary of the DE theory, it will always predict a factor of safety equal to or less than the DE theory, as can be seen in the table.

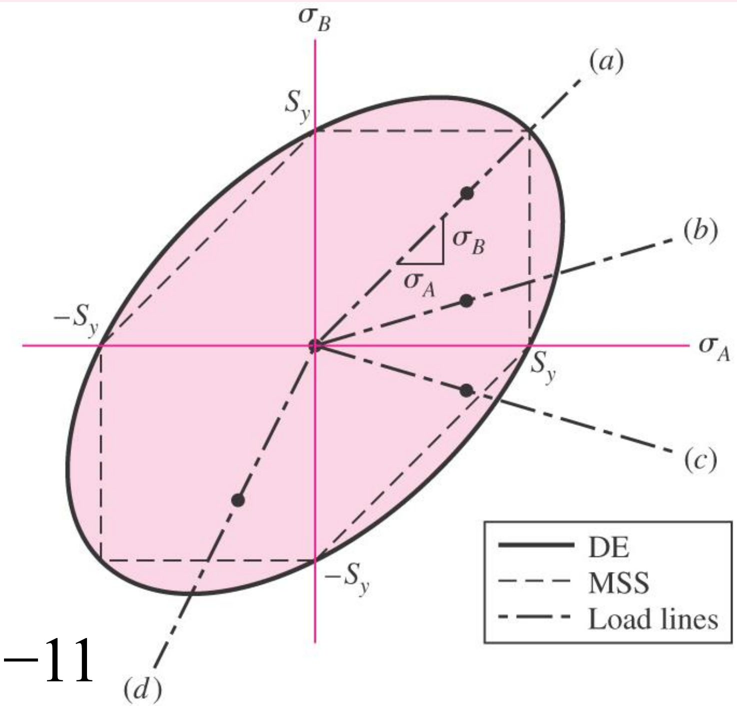


Fig. 5−11

Mohr Theory

- Some materials have compressive strengths different from tensile strengths
- *Mohr theory* is based on three simple tests: tension, compression, and shear
- Plotting Mohr's circle for each, bounding curve defines failure envelope

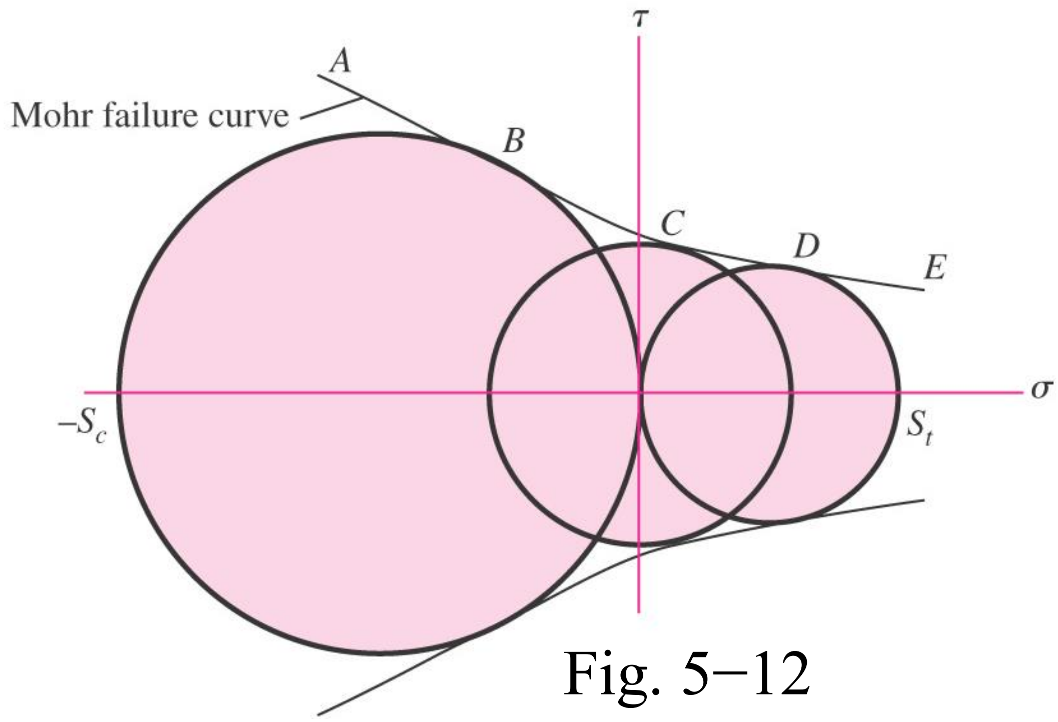


Fig. 5−12

Coulomb-Mohr Theory

- Curved failure curve is difficult to determine analytically
- *Coulomb-Mohr theory* simplifies to linear failure envelope using only tension and compression tests (dashed circles)

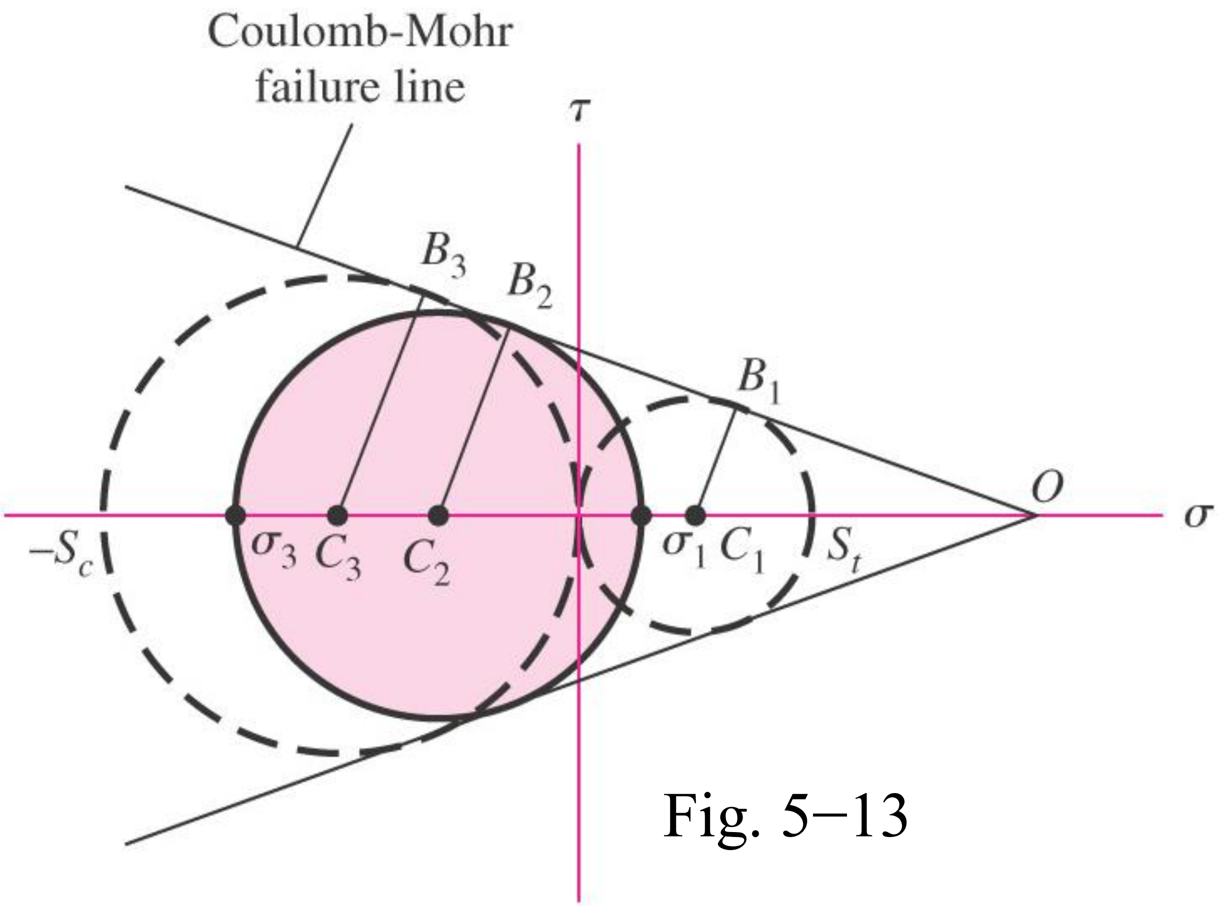


Fig. 5−13

Coulomb-Mohr Theory

- From the geometry, derive the failure criteria

$$\frac{B_2C_2 - B_1C_1}{OC_2 - OC_1} = \frac{B_3C_3 - B_1C_1}{OC_3 - OC_1}$$

$$\frac{B_2C_2 - B_1C_1}{C_1C_2} = \frac{B_3C_3 - B_1C_1}{C_1C_3}$$

$B_1C_1 = S_t/2$, $B_2C_2 = (\sigma_1 - \sigma_3)/2$, and $B_3C_3 = S_c/2$

$$\frac{\dfrac{\sigma_1 - \sigma_3}{2} - \dfrac{S_t}{2}}{\dfrac{S_t}{2} - \dfrac{\sigma_1 + \sigma_3}{2}} = \frac{\dfrac{S_c}{2} - \dfrac{S_t}{2}}{\dfrac{S_t}{2} + \dfrac{S_c}{2}}$$

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1 \qquad (5\text{–}22)$$

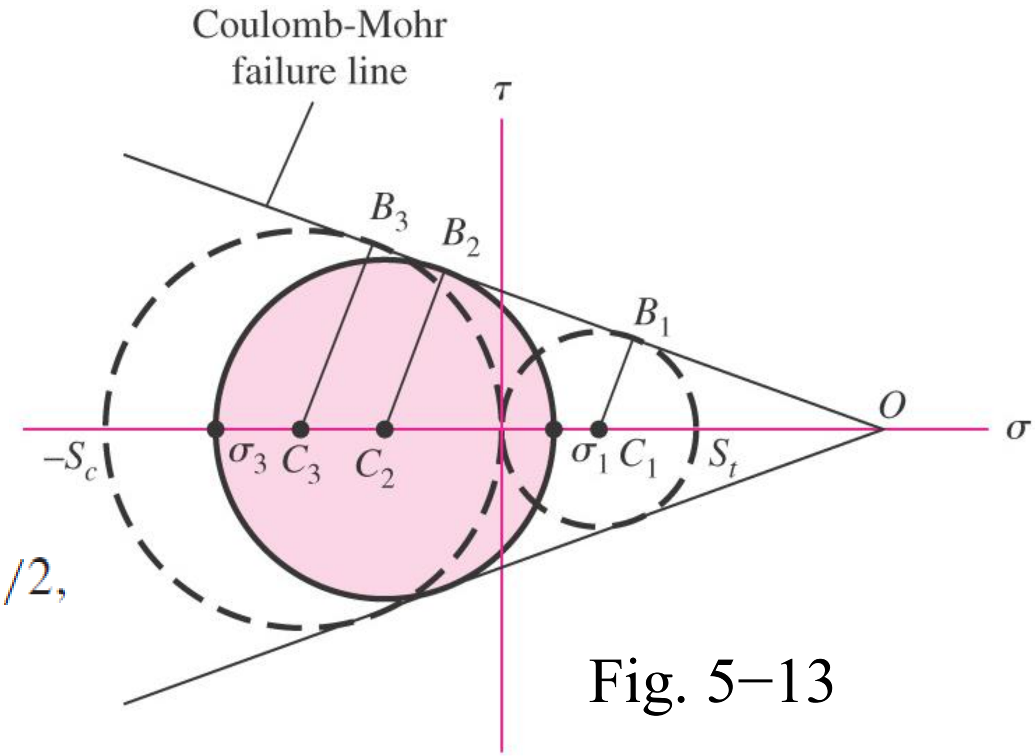


Fig. 5−13

Coulomb-Mohr Theory

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1 \qquad (5\text{–}22)$$

- To plot on principal stress axes, consider three cases
- Case 1: $\sigma_A \geq \sigma_B \geq 0$ For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$
  - Eq. (5−22) reduces to

$$\sigma_A \geq S_t \qquad (5\text{–}23)$$

- Case 2: $\sigma_A \geq 0 \geq \sigma_B$ For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = \sigma_B$
  - Eq. (5-22) reduces to

$$\frac{\sigma_A}{S_t} - \frac{\sigma_B}{S_c} \geq 1 \qquad (5\text{–}24)$$

- Case 3: $0 \geq \sigma_A \geq \sigma_B$ For this case, $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$
  - Eq. (5−22) reduces to

$$\sigma_B \leq -S_c \qquad (5\text{–}25)$$

Coulomb-Mohr Theory

- Plot three cases on principal stress axes
- Similar to MSS theory, except with different strengths for compression and tension

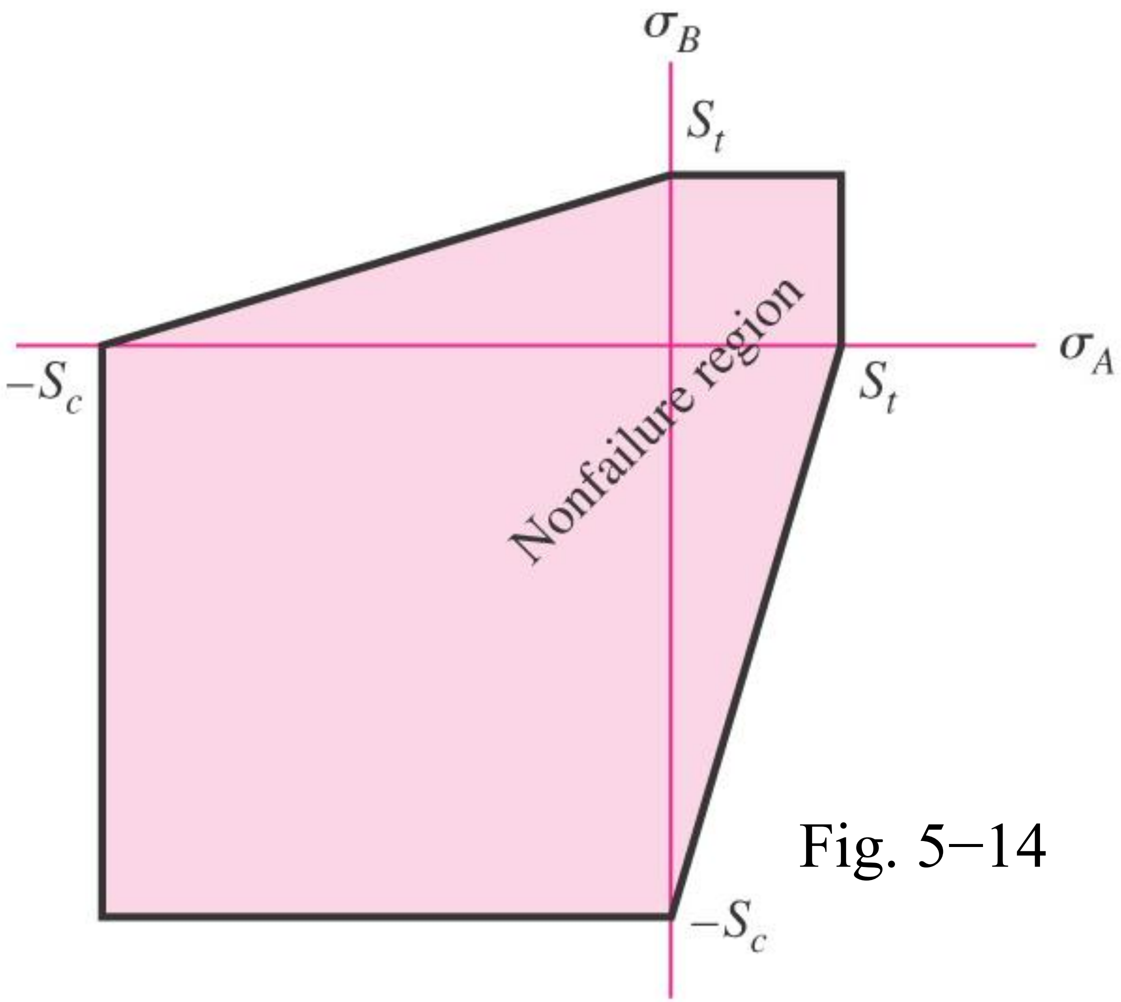


Fig. 5−14

Coulomb-Mohr Theory

- Incorporating factor of safety

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = \frac{1}{n} \qquad (5\text{–}26)$$

- For ductile material, use tensile and compressive yield strengths
- For brittle material, use tensile and compressive ultimate strengths

Coulomb-Mohr Theory

- Intersect the pure shear load line with the failure line to determine the shear strength
- Since failure line is a function of tensile and compressive strengths, shear strength is also a function of these terms.

$$S_{sy} = \frac{S_{yt} S_{yc}}{S_{yt} + S_{yc}} \qquad (5\text{–}27)$$

Example 5-2

A 25-mm-diameter shaft is statically torqued to 230 N · m. It is made of cast 195-T6 aluminum, with a yield strength in tension of 160 MPa and a yield strength in compression of 170 MPa. It is machined to final diameter. Estimate the factor of safety of the shaft.

Solution

The maximum shear stress is given by

$$\tau = \frac{16T}{\pi d^3} = \frac{16(230)}{\pi\left[25\left(10^{-3}\right)\right]^3} = 75\left(10^6\right)\,\mathrm{N/m^2} = 75\,\mathrm{MPa}$$

The two nonzero principal stresses are 75 and −75 MPa, making the ordered principal stresses $\sigma_1 = 75$, $\sigma_2 = 0$, and $\sigma_3 = -75$ MPa. From Eq. (5–26), for yield,

$$n = \frac{1}{\sigma_1/S_{yt} - \sigma_3/S_{yc}} = \frac{1}{75/160 - (-75)/170} = 1.10$$

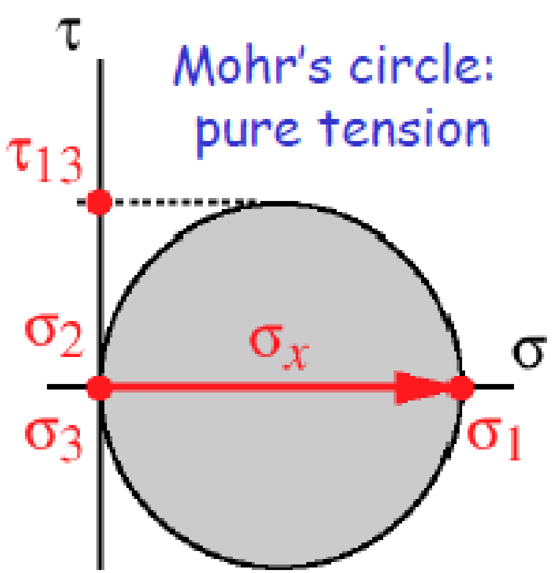
Example 5-2

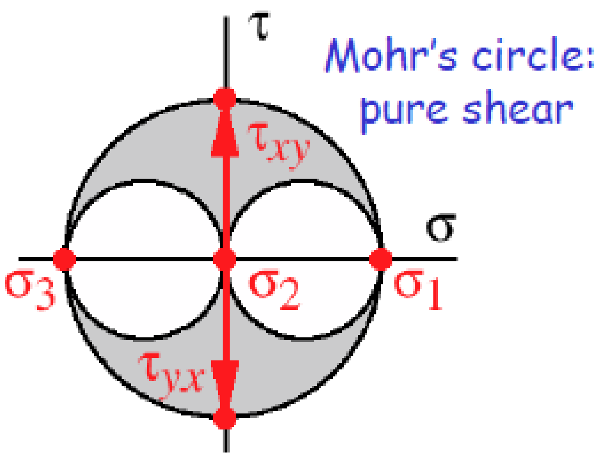
Alternatively, from Eq. (5–27),

$$S_{sy} = \frac{S_{yt} S_{yc}}{S_{yt} + S_{yc}} = \frac{160(170)}{160 + 170} = 82.4 \text{ MPa}$$

and $\tau_{\max} = 75$ MPa. Thus,

$$n = \frac{S_{sy}}{\tau_{\max}} = \frac{82.4}{75} = 1.10$$





Example 5-3

A certain force F applied at D near the end of the 15-in lever shown in Fig. 5–16, which is quite similar to a socket wrench, results in certain stresses in the cantilevered bar $OABC$. This bar ($OABC$) is of AISI 1035 steel, forged and heat-treated so that it has a minimum (ASTM) yield strength of 81 kpsi. We presume that this component would be of no value after yielding. Thus the force F required to initiate yielding can be regarded as the strength of the component part. Find this force.

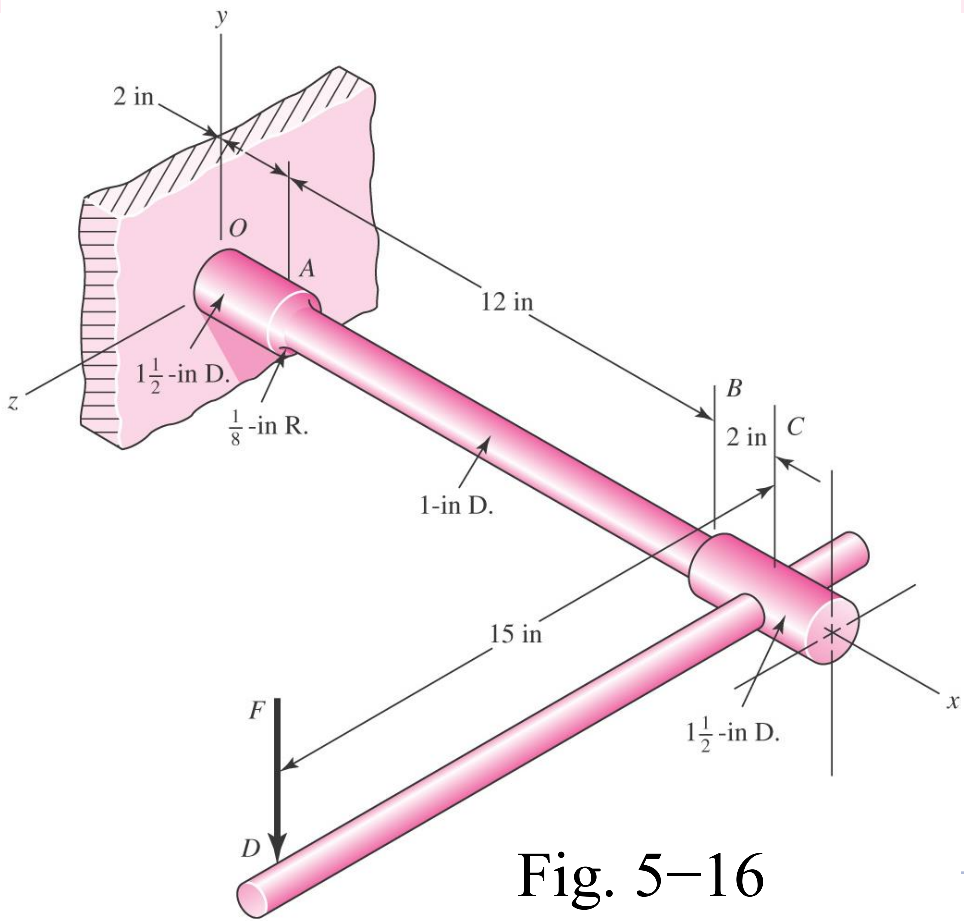


Fig. 5−16

Example 5-3

We will assume that lever *DC* is strong enough and hence not a part of the problem. A 1035 steel, heat-treated, will have a reduction in area of 50 percent or more and hence is a ductile material at normal temperatures. This also means that stress concentration at shoulder *A* need not be considered. A stress element at *A* on the top surface will be subjected to a tensile bending stress and a torsional stress. This point, on the 1-in-diameter section, is the weakest section, and governs the strength of the assembly. The two stresses are

$$\sigma_x = \frac{M}{I/c} = \frac{32M}{\pi d^3} = \frac{32(14F)}{\pi(1^3)} = 142.6F$$

$$\tau_{zx} = \frac{Tr}{J} = \frac{16T}{\pi d^3} = \frac{16(15F)}{\pi(1^3)} = 76.4F$$

Employing the distortion-energy theory, we find, from Eq. (5–15), that

$$\sigma' = \left(\sigma_x^2 + 3\tau_{zx}^2\right)^{1/2} = [(142.6F)^2 + 3(76.4F)^2]^{1/2} = 194.5F$$

Equating the von Mises stress to S_y, we solve for F and get

$$F = \frac{S_y}{194.5} = \frac{81\ 000}{194.5} = 416\ \text{lbf}$$

Example 5-3

In this example the strength of the material at point A is $S_y = 81$ kpsi. The strength of the assembly or component is $F = 416$ lbf.

Let us apply the MSS theory for comparison. For a point undergoing plane stress with only one nonzero normal stress and one shear stress, the two nonzero principal stresses will have opposite signs, and hence the maximum shear stress is obtained from the Mohr's circle between them. From Eq. (3–14)

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{zx}^2} = \sqrt{\left(\frac{142.6F}{2}\right)^2 + (76.4F)^2} = 104.5F$$

Setting this equal to $S_y/2$, from Eq. (5–3) with $n = 1$, and solving for F, we get

$$F = \frac{81\,000/2}{104.5} = 388 \text{ lbf}$$

which is about 7 percent less than found for the DE theory. As stated earlier, the MSS theory is more conservative than the DE theory.

Example 5-4

The cantilevered tube shown in Fig. 5–17 is to be made of 2014 aluminum alloy treated to obtain a specified minimum yield strength of 276 MPa. We wish to select a stock-size tube from Table A–8 using a design factor $n_d = 4$. The bending load is $F = 1.75$ kN, the axial tension is $P = 9.0$ kN, and the torsion is $T = 72$ N · m. What is the realized factor of safety?

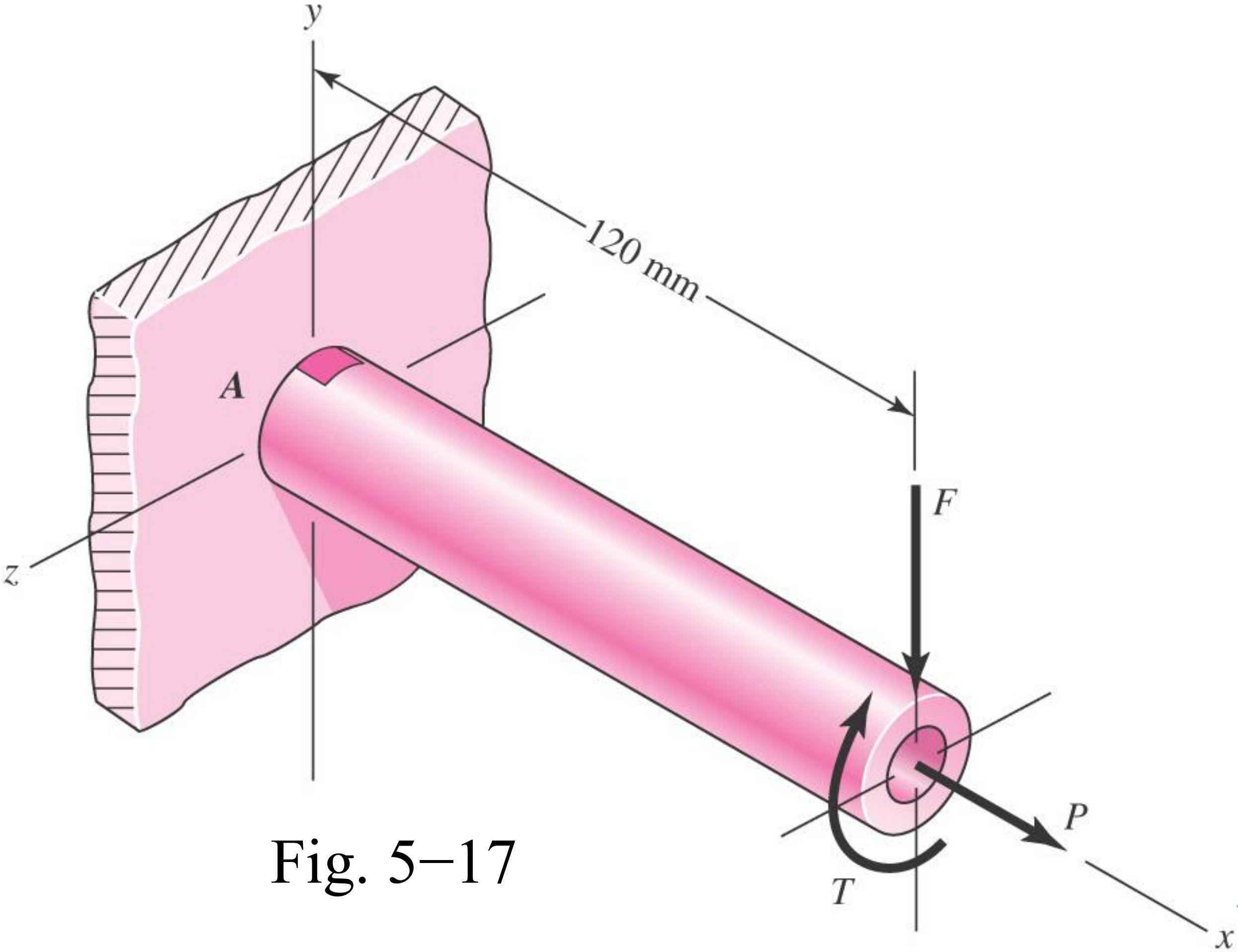


Fig. 5−17

m = unit mass, kg/m
A = area, in^2 (cm^2)
I = second moment of area, in^4 (cm^4)
J = second polar moment of area, in^4 (cm^4)
k = radius of gyration, in (cm)
Z = section modulus, in^3 (cm^3)
d, t = size (OD) and thickness, in (mm)

Table A–8

Properties of Round Tubing

Size, mm	*m*	*A*	*I*	*k*	*Z*	*J*
12 × 2	0.490	0.628	0.082	0.361	0.136	0.163
16 × 2	0.687	0.879	0.220	0.500	0.275	0.440
16 × 3	0.956	1.225	0.273	0.472	0.341	0.545
20 × 4	1.569	2.010	0.684	0.583	0.684	1.367
25 × 4	2.060	2.638	1.508	0.756	1.206	3.015
25 × 5	2.452	3.140	1.669	0.729	1.336	3.338
30 × 4	2.550	3.266	2.827	0.930	1.885	5.652
30 × 5	3.065	3.925	3.192	0.901	2.128	6.381
42 × 4	3.727	4.773	8.717	1.351	4.151	17.430
42 × 5	4.536	5.809	10.130	1.320	4.825	20.255
50 × 4	4.512	5.778	15.409	1.632	6.164	30.810
50 × 5	5.517	7.065	18.118	1.601	7.247	36.226

Example 5-4

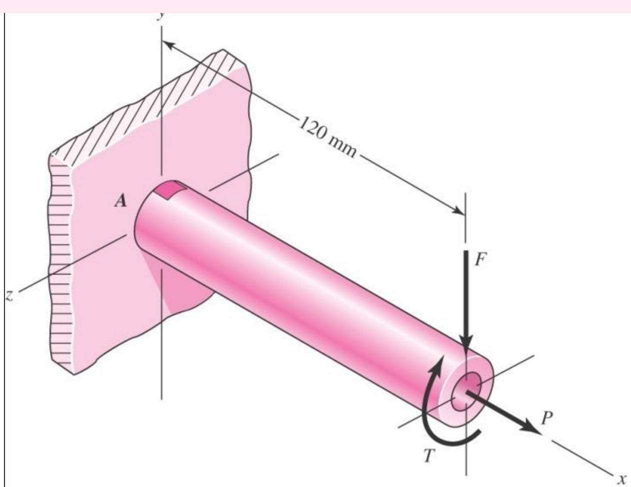
The critical stress element is at point A on the top surface at the wall, where the bending moment is the largest, and the bending and torsional stresses are at their maximum values. The critical stress element is shown in Fig. 5–17b. Since the axial stress and bending stress are both in tension along the x axis, they are additive for the normal stress, giving

$$\sigma_x = \frac{P}{A} + \frac{Mc}{I} = \frac{9}{A} + \frac{120(1.75)(d_o/2)}{I} = \frac{9}{A} + \frac{105 d_o}{I} \quad (1)$$

where, if millimeters are used for the area properties, the stress is in gigapascals.

The torsional stress at the same point is

$$\tau_{zx} = \frac{Tr}{J} = \frac{72(d_o/2)}{J} = \frac{36 d_o}{J} \quad (2)$$



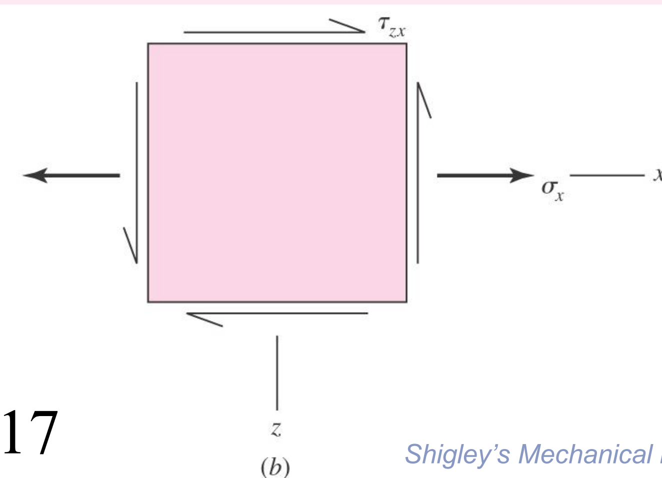


Fig. 5−17

Example 5-4

For accuracy, we choose the distortion-energy theory as the design basis. The von Mises stress from Eq. (5–15), is

$$\sigma' = \left(\sigma_x^2 + 3\tau_{zx}^2\right)^{1/2} \tag{3}$$

On the basis of the given design factor, the goal for σ' is

$$\sigma' \leq \frac{S_y}{n_d} = \frac{0.276}{4} = 0.0690 \text{ GPa} \tag{4}$$

where we have used gigapascals in this relation to agree with Eqs. (1) and (2).

Example 5-4

Programming Eqs. (1) to (3) on a spreadsheet and entering metric sizes from Table A–8 reveals that a 42 × 5-mm tube is satisfactory. The von Mises stress is found to be $\sigma' = 0.06043$ GPa for this size. Thus the realized factor of safety is

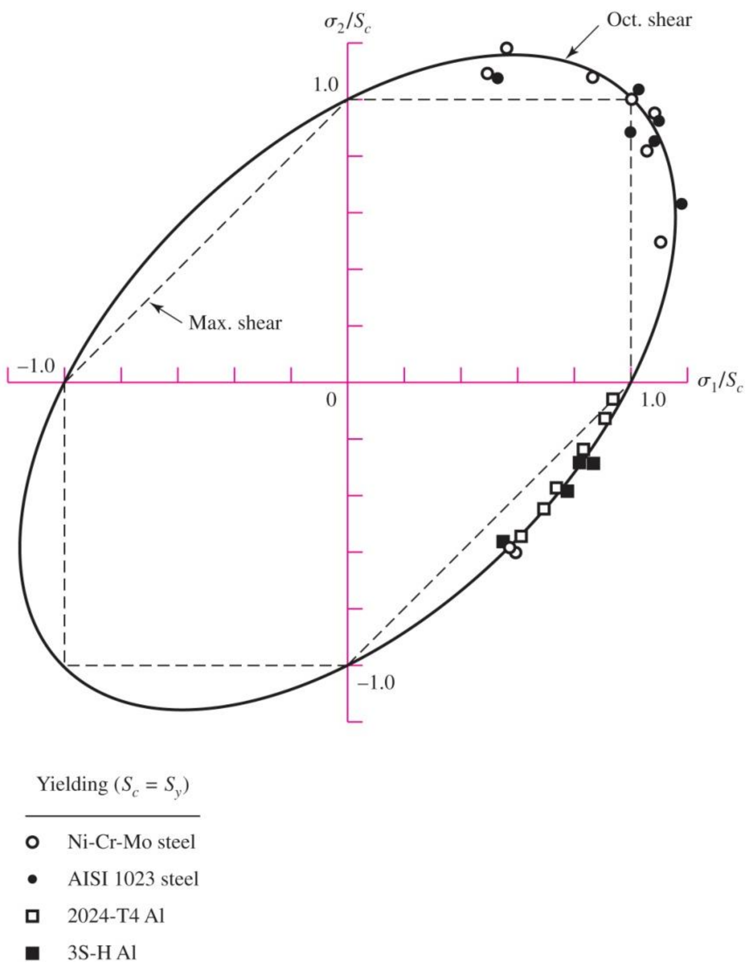
$$n = \frac{S_y}{\sigma'} = \frac{0.276}{0.06043} = 4.57$$

For the next size smaller, a 42 × 4-mm tube, $\sigma' = 0.07105$ GPa giving a factor of safety of

$$n = \frac{S_y}{\sigma'} = \frac{0.276}{0.07105} = 3.88$$

Failure of Ductile Materials Summary

- Either the maximum-shear-stress theory or the distortion-energy theory is acceptable for design and analysis of materials that would fail in a ductile manner.
- For design purposes the maximum-shear-stress theory is easy, quick to use, and conservative.
- If the problem is to learn why a part failed, then the distortion-energy theory may be the best to use.
- For ductile materials with unequal yield strengths, S_{yt} in tension and S_{yc} in compression, the Mohr theory is the best available.

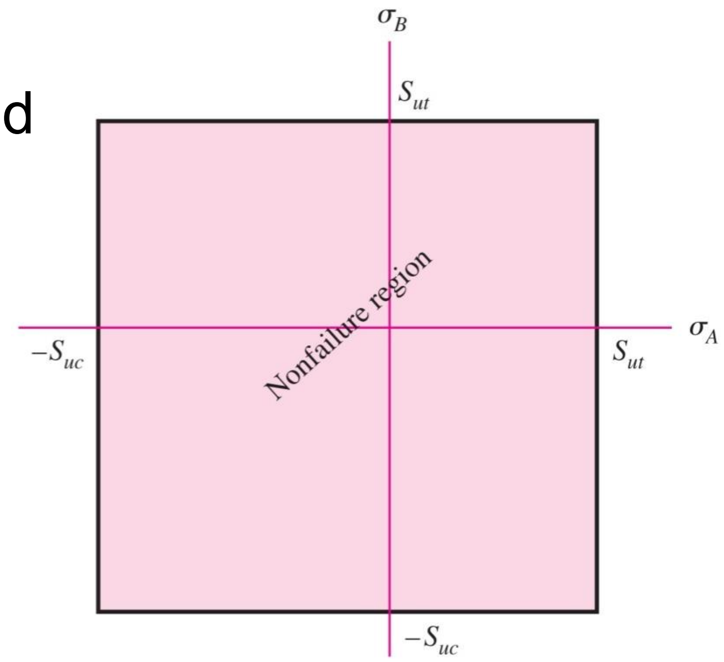


Maximum-Normal-Stress Theory for Brittle Materials

- The maximum-normal-stress (MNS) theory states that **failure occurs whenever one of the three principal stresses equals or exceeds the ultimate strength.**
- For a general stress state in the ordered form $\sigma_1 \geq \sigma_2 \geq \sigma_3$. This theory then predicts that failure occurs whenever

$$\sigma_1 \geq S_{ut} \qquad \text{or} \qquad \sigma_3 \leq -S_{uc}$$

where S_{ut} and S_{uc} are the ultimate tensile and compressive strengths, respectively, given as positive quantities.



- MNS theory is not very good at predicting failure in the fourth quadrant of the σ_A, σ_B plane. Hence not recommended for use (*has been added for historical reason*!)

Maximum Normal Stress Theory

- Theory: Failure occurs when the maximum principal stress in a stress element exceeds the strength.
- Predicts failure when

$$\sigma_1 \geq S_{ut} \qquad \text{or} \qquad \sigma_3 \leq -S_{uc} \tag{5–28}$$

- For plane stress,

$$\sigma_A \geq S_{ut} \qquad \text{or} \qquad \sigma_B \leq -S_{uc} \tag{5–29}$$

- Incorporating design factor,

$$\sigma_A = \frac{S_{ut}}{n} \qquad \text{or} \qquad \sigma_B = -\frac{S_{uc}}{n} \tag{5–30}$$

Brittle Coulomb-Mohr

- Same as previously derived, using ultimate strengths for failure
- Failure equations dependent on quadrant

Quadrant condition	Failure criteria	
$\sigma_A \ge \sigma_B \ge 0$	$\sigma_A = \dfrac{S_{ut}}{n}$	(5–31*a*)
$\sigma_A \ge 0 \ge \sigma_B$	$\dfrac{\sigma_A}{S_{ut}} - \dfrac{\sigma_B}{S_{uc}} = \dfrac{1}{n}$	(5–31*b*)
$0 \ge \sigma_A \ge \sigma_B$	$\sigma_B = -\dfrac{S_{uc}}{n}$	(5–31*c*)

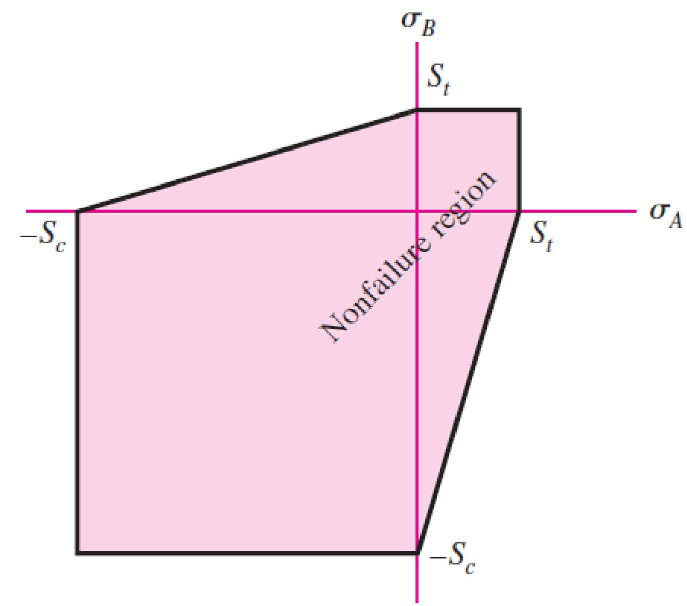


Fig. 5−14

Brittle Failure Experimental Data

- Coulomb-Mohr is conservative in 4th quadrant
- *Modified Mohr* criteria adjusts to better fit the data in the 4th quadrant

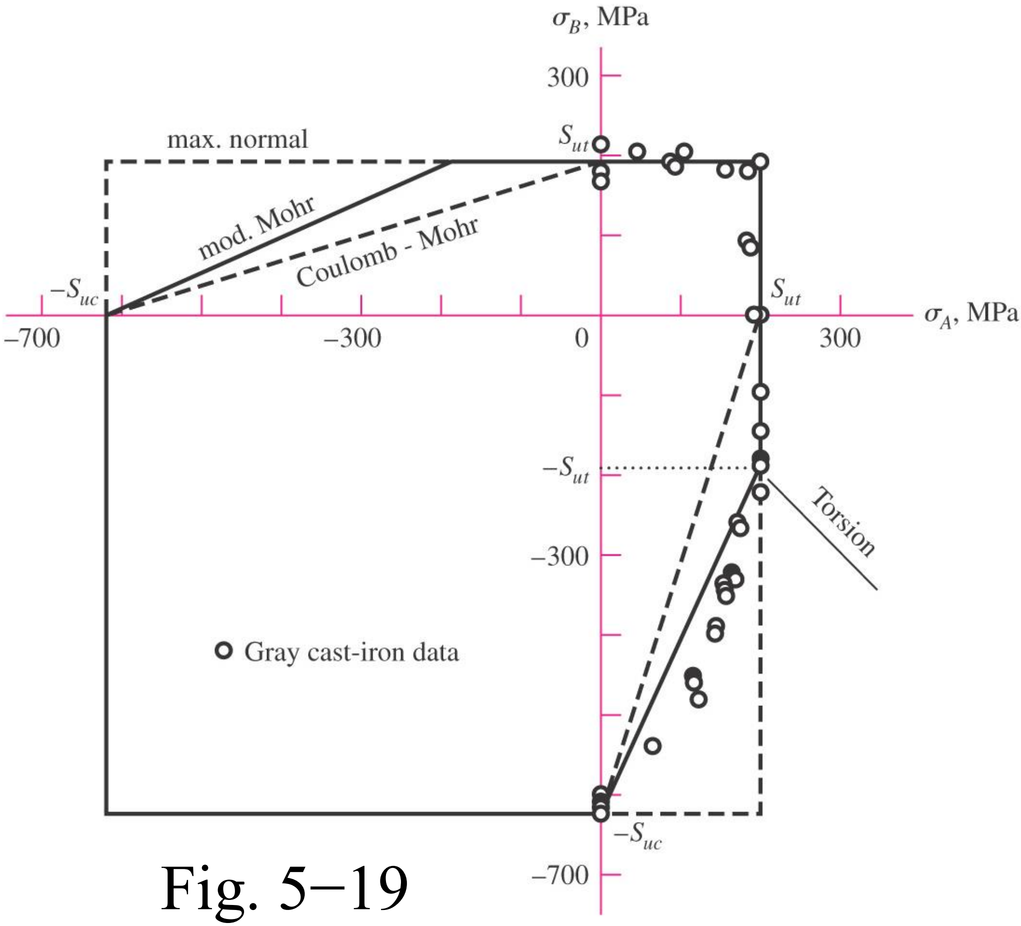
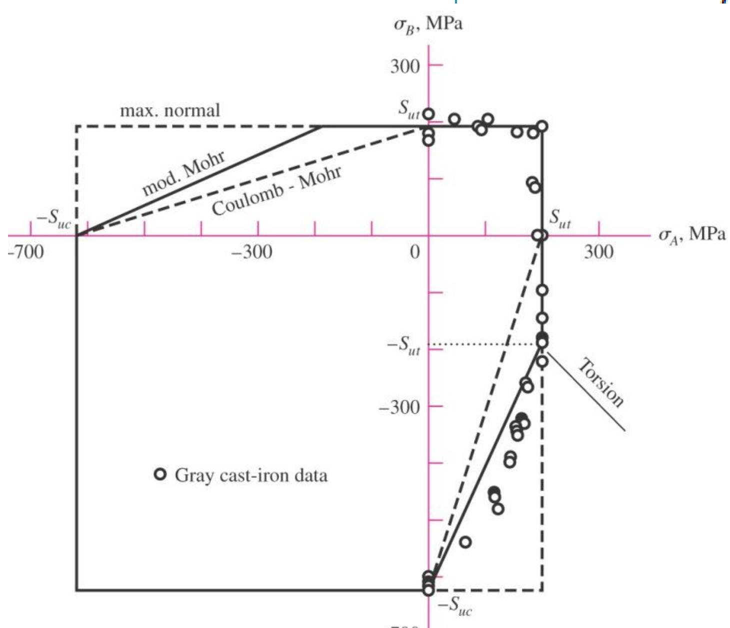


Fig. 5–19

Modified-Mohr

Quadrant condition	Failure criteria	
$\sigma_A \ge \sigma_B \ge 0$	$\sigma_A = \dfrac{S_{ut}}{n}$	(5–32*a*)
$\sigma_A \ge 0 \ge \sigma_B$ and $\left\lvert \dfrac{\sigma_B}{\sigma_A} \right\rvert \le 1$	$\sigma_A = \dfrac{S_{ut}}{n}$	(5–32*a*)
$\sigma_A \ge 0 \ge \sigma_B$ and $\left\lvert \dfrac{\sigma_B}{\sigma_A} \right\rvert > 1$	$\dfrac{(S_{uc} - S_{ut})\,\sigma_A}{S_{uc} S_{ut}} - \dfrac{\sigma_B}{S_{uc}} = \dfrac{1}{n}$	(5–32*b*)
$0 \ge \sigma_A \ge \sigma_B$	$\sigma_B = -\dfrac{S_{uc}}{n}$	(5–32*c*)



Example 5-5

Consider the wrench in Ex. 5–3, Fig. 5–16, as made of cast iron, machined to dimension. The force F required to fracture this part can be regarded as the strength of the component part. If the material is ASTM grade 30 cast iron, find the force F with
(a) Coulomb-Mohr failure model.
(b) Modified Mohr failure model.

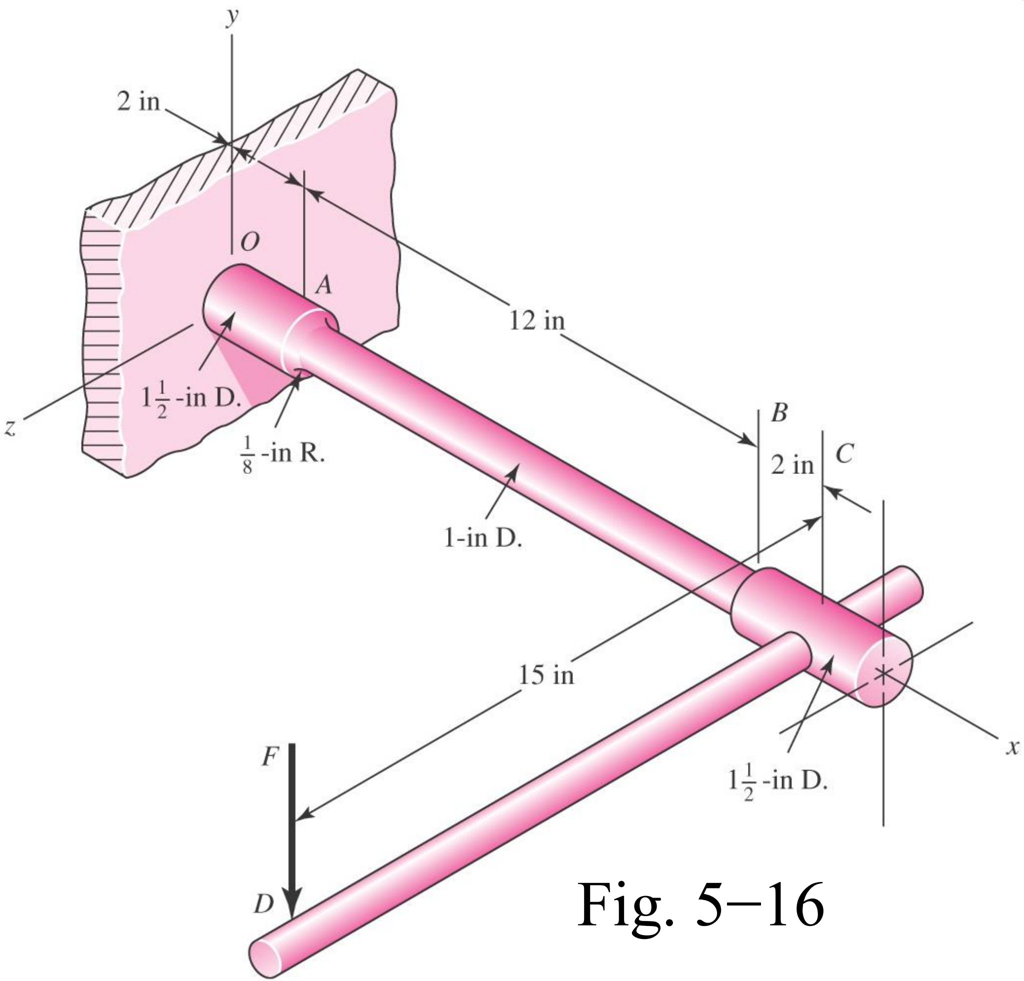


Fig. 5−16

Example 5-5

We assume that the lever *DC* is strong enough, and not part of the problem. Since grade 30 cast iron is a brittle material *and* cast iron, the stress-concentration factors K_t and K_{ts} are set to unity. From Table A–24, the tensile ultimate strength is 31 kpsi and the compressive ultimate strength is 109 kpsi. The stress element at *A* on the top surface will be subjected to a tensile bending stress and a torsional stress. This location, on the 1-in-diameter section fillet, is the weakest location, and it governs the strength of the assembly. The normal stress σ_x and the shear stress at *A* are given by

Table A24, P1046

$$\sigma_x = K_t \frac{M}{I/c} = K_t \frac{32M}{\pi d^3} = (1)\frac{32(14F)}{\pi(1)^3} = 142.6F$$

$$\tau_{xy} = K_{ts} \frac{Tr}{J} = K_{ts} \frac{16T}{\pi d^3} = (1)\frac{16(15F)}{\pi(1)^3} = 76.4F$$

From Eq. (3–13) the nonzero principal stresses σ_A and σ_B are

$$\sigma_A, \sigma_B = \frac{142.6F + 0}{2} \pm \sqrt{\left(\frac{142.6F - 0}{2}\right)^2 + (76.4F)^2} = 175.8F, -33.2F$$

This puts us in the fourth-quadrant of the σ_A, σ_B plane.

Example 5-5

(*a*) For BCM, Eq. (5–31*b*) applies with $n = 1$ for failure.

$$\frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{175.8F}{31(10^3)} - \frac{(-33.2F)}{109(10^3)} = 1$$

Solving for F yields

$$F = 167 \text{ lbf}$$

(*b*) For MM, the slope of the load line is $|\sigma_B/\sigma_A| = 33.2/175.8 = 0.189 < 1$. Obviously, Eq. (5–32*a*) applies.

$$\frac{\sigma_A}{S_{ut}} = \frac{175.8F}{31(10^3)} = 1$$

$$F = 176 \text{ lbf}$$

As one would expect from inspection of Fig. 5–19, Coulomb-Mohr is more conservative.

Selection of Failure Criteria

- First determine ductile vs. brittle
- For ductile
  - MSS is conservative, often used for design where higher reliability is desired
  - DE is typical, often used for analysis where agreement with experimental data is desired
  - If tensile and compressive strengths differ, use Ductile Coulomb-Mohr
- For brittle
  - Mohr theory is best, but difficult to use
  - Brittle Coulomb-Mohr is very conservative in 4^{th} quadrant
  - Modified Mohr is still slightly conservative in 4^{th} quadrant, but closer to typical

Selection of Failure Criteria in Flowchart Form

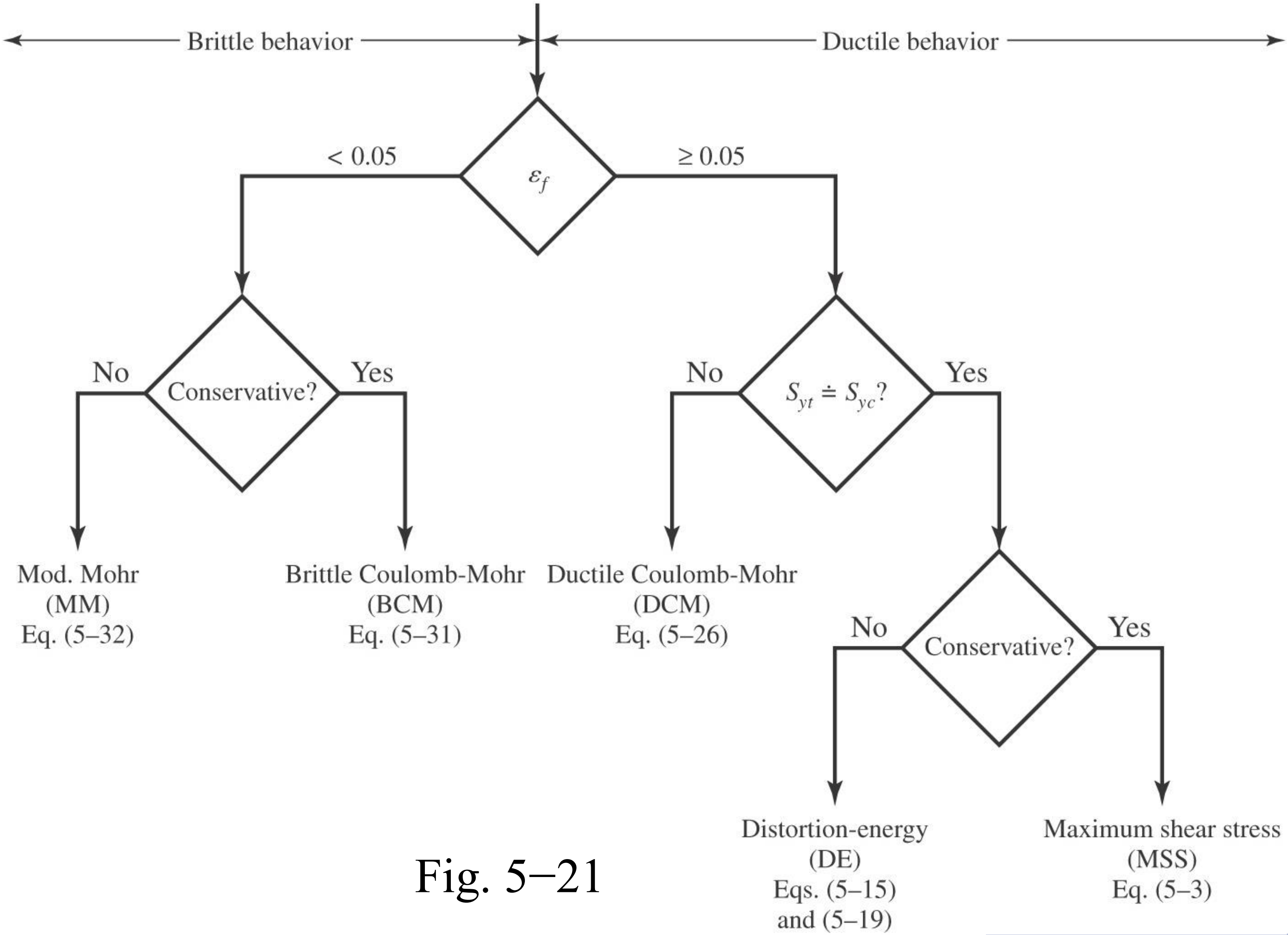


Fig. 5−21